1. Find an example of a sequence of random variables $(X_n)_{n=1}^\infty$ and a random variable X, such that $\mathbb{E}X_n = n$ for each $n \in \mathbb{N} = \{1, 2, \dots\}$, and as $n \to \infty$, X_n converges to X in probability but not almost surely.

2. Let $(X_k)_{k=1}^\infty$ be a sequence of independent random variables with distributions
$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = 2) = \frac{1}{4}, \quad \mathbb{P}(X_k = 3k^2) = \frac{1}{2} - \mathbb{P}(X_k = 0) = \frac{1}{2k^2},$$
for each $k \in \mathbb{N}$.

   (a) Set $Y_k = X_k \mathbb{1}_{\{|X_k| \le 2\}}, k \in \mathbb{N}$. Prove that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Y_k$$
exists almost surely.

   (b) Set
$$S_n = \sum_{k=0}^n X_k, \quad n \in \mathbb{N}.$$
Does $\lim_{n\to\infty} \frac{1}{n} S_n$ exist almost surely? If yes, give the value of the limit and justify your answer. If your answer is no, explain why. You may use the statement in part (a) directly.

3. Suppose $(X_k)_{k=1}^\infty$ are independent random variables with
$$\mathbb{P}(X_k = 2k) = 1 - \mathbb{P}(X_k = 0) = \frac{1}{2k}, \quad k \in \mathbb{N}.$$
Set $S_n = X_1 + \cdots + X_n$. Show that there exist a_n, b_n such that
$$\frac{S_n - a_n}{b_n} \Rightarrow \mathcal{N}(0, 1),$$
as $n \to \infty$, where the right-hand side denote a standard Gaussian random variable. Justify your answer and provide explicit expressions for a_n, b_n.

4. Consider triangular array of random variables $(X_{n,k})_{n\in\mathbb{N},k=1,\dots,n}$, and assume that for each n, $(X_{n,k})_{k=1,\dots,n}$ are i.i.d. exponential random variables with $\mathbb{E}X_{n,1} = 1/n$. Set $b_n = (\log n)/n$. Compute
$$\lim_{n\to\infty} \mathbb{P}\left(\max_{k=1,\dots,n} X_{n,k} \le b_n\right).$$