In this exam $\mathbb{R}$ denotes the field of all real numbers; $\mathbb{R}^d$ is the d-dimensional Euclidean space with the usual norm $\|x\| = \left(\sum_{k=1}^{d} x_k^2\right)^{1/2}$; $C[0,1]$ is the space of continuous functions on the interval $[0,1]$. Proofs or counterexamples are required for all problems.

1. If f is continuous on $[a,b]$, if $a < c < d < b$, and $M = f(c) + f(d)$, prove that there exists a number ξ between a and b such that $M = 2f(\xi)$.

2. Prove that if a set C in $\mathbb{R}^d$ is connected and a point $x \in \mathbb{R}^d$ is a cluster point of C, then the set $C \cup \{x\}$ is connected.

3. Prove the Monotone Convergence Theorem for Sequences as stated below. Note: For the "only if" part, do not simply state that a convergent sequence is bounded; prove it.

   Let $\{x_n\}$ be a monotone increasing sequence of real numbers. Then $\{x_n\}$ is convergent if and only if it is bounded.

4. Prove or give a counterexample: Let f and g be two functions on the interval $[-1,1]$. If the product fg is Riemann integrable on $[-1,1]$, then at least one of f and g must be Riemann integrable on $[-1,1]$. Carefully support all your statements.

5. Let $X = \{f \in C[0,1] : f(0) = 0\}$. You may assume that X is a vector space over $\mathbb{R}$. For each $f \in X$, let $(Tf)(x) = \int_0^x f(y)\,dy,\ x \in [0,1]$.
   (a) Show that T is a linear map from X to itself.
   (b) Show that T is injective.
   (c) Show that T is not surjective.

6. Let V be a vector space, and $T\colon V \to V$ a linear map. Suppose there exist linearly independent vectors v_1, v_2, v_3 such that $Tv_1 = v_2$, $Tv_2 = v_3$, and $Tv_3 = v_2$. Show that $\lambda = 0$, $\lambda = 1$, and $\lambda = -1$ are eigenvalues of T. (Hint: consider appropriate linear combinations of v_1, v_2, and v_3 as possible eigenvectors.)

7. Let ℓ^2 be the set of all real sequences $\{a_n\}_1^\infty$ such that $\sum_{n=1}^\infty |a_n|^2 < \infty$. Prove that ℓ^2 is a vector space over $\mathbb{R}$ and that $\langle \{a_n\}, \{b_n\} \rangle := \sum_{n=1}^\infty a_n b_n$ defines an inner product on ℓ^2.

8. Let V be a finite-dimensional vector space and T a linear map from V to itself. Suppose range $(T - 2I) \subseteq$ null $(T - 3I)$. Show that T is invertible.