Statistical Methods Preliminary Exam

August 2023

There are 4 problems with a total of 100 points.

Show all of your work.

1. (25 points) Let a random variable X have the binomial distribution $b(p,n)$, and let the function $g(p)$ defined as $g(p) = p(1-p)$.

   (a) Show that the UMVUE of $g(p)$ is $\hat{\delta} = X(n-X)/n(n-1)$.
   (UMVUE: uniformly minimum variance unbiased estimator).

   (b) Determine the limiting distribution of $\sqrt{n}(\hat{\delta} - g(p))$ and $n(\hat{\delta} - g(p))$ when $g'(p) \neq 0$ and $g'(p) = 0$, respectively.

2. (25 points) Let the random variable X follow the *inverse Gaussian* distribution $I(\mu, \tau)$ with density

$$\sqrt{\frac{\tau}{2\pi x^3}} \exp\left(-\frac{\tau}{2x\mu^2}(x-\mu)^2\right), \quad x > 0, \ \ \tau, \mu > 0.$$

   (a) Find the moment generating function of X.

   (b) Show that $V = \frac{\tau}{X\mu^2}(X-\mu)^2 \sim \chi_1^2$

   Let $X_1, \ldots, X_n$ be a random sample from $I(\mu, \tau)$.

   (c) Show that $\overline{X} = \sum_{i=1}^n X_i/n \sim I(\mu, n\tau)$.

   (d) Show that there exists a UMP test for testing $H_0 : \mu \leq \mu_0$ versus. $H_1 : \mu > \mu_0$ when τ is known. (UMP: uniformly most powerful).

3. (25 points) Let $X_1, X_2, \ldots, X_m$ be a random sample from an exponential distribution with mean λ, and $Y_1, Y_2, \ldots, Y_n$ be a random sample form an exponential distribution with mean μ, and assume that the two samples are independent.

   (a) Find the LRT statistic, T, for testing the null hypothesis $H_0 : \lambda = \mu$ versus the alternative hypothesis $H_1 : \lambda \neq \mu$. (LRT: Likelihood Ratio Test).

   (b) Using a suitable one-to-one transformation of T, find the exact 5% critical region for the LRT in (a). Give the critical region in terms of the percentile(s) of a known distribution. Clearly identify the distribution and which percentiles(s), upper or lower.

4. (25 points) Let $X_1, \ldots, X_n$ be a random sample from $U(\theta, \theta+1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for θ given by the probability density function, for $-\infty < \theta < \infty$,

$$\pi(\theta) = \frac{1}{2}e^{-|\theta|} \ .$$

   (a) Find the posterior distribution of θ, given $(X_1 = x_1, \ldots, X_n = x_n)$, i.e., $\pi(\theta | x_1, \cdots, x_n)$.

   (b) Find the Bayes estimator of θ under the loss function, $L(\theta, \delta) = (\theta - \delta)^2$.