Preliminary Exam

Complex Analysis August 2024

Please start each problem on a new page, and arrange your work in order when submitting.
Please choose 5 problems out of the 6 problems on the exam. Only 5 problems will be graded.

In this exam $\mathbb{C}$ denotes the complex plane.

1. Calculate the Laurent series of the function $f(z) = \dfrac{1}{(z+1)(z-2)}$ that is convergent in the annulus $\{z \in \mathbb{C}\colon 1 < |z| < 2\}$.

2. Let $\mathbb{D}$ denote the open unit disk. Show that there is no function f that is holomorphic in $\mathbb{D}$ and extends continuously to the boundary $\partial\mathbb{D}$ such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$.

3. Use the Residue Theorem to compute $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,\mathrm{d}x$.

4. Find the number of zeros (counting with multiplicity) of the function $f(z) = z^6 - 5z^2 + \sin(z)$ that lie in the open annulus $\{z \in \mathbb{C}\colon 1 < |z| < 2\}$.

5. Let $\mathbb{D}$ denote the open unit disk. Let $\varphi\colon \mathbb{D} \to \mathbb{H}$ be holomorphic, where $\mathbb{H} = \{z \in \mathbb{D}\colon \operatorname{Re}(z) > 0\}$. Suppose that $\varphi(0) = 1$ and that $\varphi'(0) = 2\mathrm{i}$. Prove that φ is a conformal map (that is it is bijective with a holomorphic inverse).

6. Let f be a holomorphic function on $\Omega = \{z \in \mathbb{C} : |z-2| \leqslant 1\}$ such that $|f(z)| \leqslant 2|z|^2$ for $z \in \partial\Omega$. Prove that $|f(2)| \leqslant 8$.