Ordinary Differential Equation Preliminary Exam

August 2024

Answer any four out of the five questions.

Problem 1 Let $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

a. For the system $\dot{x}(t) = Ax(t)$ identify the rest point at the origin $(0,0)$. Sketch the integral curves near the origin, and indicate the direction in which they are traveled for increasing t.

b. Calculate e^{At}.

c. Solve the system

$$\dot{x} = Ax, \quad x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Problem 2 Consider a *Hamiltonian system*

$$\begin{aligned} \dot{x}_1 &= V_{x_2}(x_1, x_2) \\ \dot{x}_2 &= -V_{x_1}(x_1, x_2)\,, \end{aligned}$$

where $V(x_1, x_2)$ is a given twice differentiable function.

(i) Show that

$$V(x_1(t), x_2(t)) = \text{constant}\,,$$

for any solution $(x_1(t), x_2(t))$.

(ii) Show that a point $P = (x_1^0, x_2^0)$ is a rest point if and only if P is a critical point of $V(x_1, x_2)$.

(iii) Let (a, b) be a point of strict local minimum or maximum of $V(x_1, x_2)$. Show that (a, b) is a center.

Problem 3 The second order differential equation

$$x'' + f(x)x' + g(x) = 0$$

can be written as the Lienard system

$$\dot{x}_1 = x_2 - F(x_1), \qquad \dot{x}_2 = -g(x_1)$$

where

$$F(x_1) = \int_0^{x_1} f(s)ds.$$

Let

$$G(x_1) = \int_0^{x_1} g(s)ds$$

and suppose that $g(0) = 0$ and

$$G(x_1) > 0, \qquad g(x_1)F(x_1) > 0$$

for any $x_1 \neq 0$. Show that the origin is an asymptotically stable equilibrium point.

Problem 4 Consider the system

$$\dot{x} = -x^2 - y^2 + 1, \qquad \dot{y} = 2xy.$$

(i) Find all equilibrium points of the system.

(ii) Determine the nature of the equilibrium points; be as specific as possible.

Problem 5 Let A be a continuous family of $n \times n$ matrices and let $P(t)$ be the matrix solution to the initial value problem $\frac{\mathrm{d}P}{\mathrm{d}t} = A(t)P, P(0) = P_0$. Show that

$$det P(t) = (det P_0) exp\left(\int_0^t Tr A(s)\,\mathrm{d}s\right).$$