1. Provide an example of a sequence of independent random variables $\{X_n\}_{n\in\mathbb{N}}$ such that the following conditions hold *at the same time*.

   (a) $\mathbb{E}|X_n| = n$ for all $n \in \mathbb{N}$.

   (b) $\lim_{n\to\infty} X_n = 0$ almost surely.

   Explain why your example satisfies the two conditions.

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with

$$\mathbb{P}(X_1 > x) = e^{-x^3}, x > 0.$$

   Find a value $\beta > 0$ such that

$$\mathbb{P}\left(\frac{X_n}{(\log n)^\beta} \geq 1-\epsilon \text{ i.o.}\right) = 1, \text{ for all } \epsilon > 0,$$
$$\mathbb{P}\left(\frac{X_n}{(\log n)^\beta} \geq 1+\epsilon \text{ i.o.}\right) = 0, \text{ for all } \epsilon > 0.$$

   Justify your answer.

3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables each with parameter $1/n$. That is, $\mathbb{P}(X_n = 1) = 1/n = 1 - \mathbb{P}(X_n = 0)$. Find sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$\frac{X_1 + \cdots + X_n - b_n}{a_n}$$

   converges in distribution to a standard normal random variable, and justify your answer. You may want to recall the fact that $\lim_{n\to\infty}(\sum_{j=1}^n 1/j)/\log n = 1$.

4. Let $\{X_k\}_{k=1}^\infty$ be i.i.d. Beta random variables with parameters $\alpha > 0, \beta > 0$. That is, X_k is a continuous random variable with probability density function

$$p_{\alpha,\beta}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\mathbf{1}_{\{x\in(0,1)\}},$$

   where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \in (0,\infty)$ is the beta function. You can use $B(\alpha,\beta)$ directly in your answer (without evaluating explicitly its value).

   You may answer part (b) directly, and your answer should include part (a) as a special case.

   (a) Assume $\beta = 1$. Find a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that

$$a_n \min_{k=1,\ldots,n} X_k \tag{1}$$

   converges in distribution to a non-degenerate random variable. Identify the cumulative distribution of the limit. The sequence and the limit may depend on the value of α.

   (b) Assume $\beta > 0$. Answer the same question as in the previous part. The sequence and the limit may depend on the values of α and β.

   Hint: you may want to evaluate the following integral

$$c_n(x) := \int_0^{x/a_n} y^{\alpha-1}(1-y)^{\beta-1}dy,$$

   and you want to show that for all $x > 0$, by appropriately choosing the rate $a_n \to \infty$, $\lim_{n\to\infty} nc_n(x) = c(x)$ for some non-constant function $c(x)$. For this purpose, you may first apply a change of variables to the integral expression of $c_n(x)$ and then apply the dominated convergence theorem.