Real Analysis Preliminary Exam, August 22, 2024

Time allowed: 2 hours 30 minutes.

This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted.

Notation: $\mathbb{R}$ is the set of real numbers; m is the Lebesgue measure on $\mathbb{R}$; m^* is the Lebesgue outer measure on $\mathbb{R}$. In this exam, "measurable" means "Lebesgue measurable", and "integrable" means "Lebesgue integrable".

Helpful suggestion: When proving integrability do not forget to verify measurability.

1. Let $A \subset (0,1)$ and $B = (0,1) \setminus A$. Prove that

$$\sup \{ m^*(F) \colon F \subset A, F \text{ is closed} \} = 1 - m^*(B).$$

2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove that for any $\varepsilon > 0$ there is a measurable function $g \colon \mathbb{R} \to \mathbb{R}$ that takes on only a countable number of values, such that $|f(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$.

3. Let f and g be non-negative measurable functions on $[0,1]$ such that $\int_{[0,1]} f \, dm \leq 1$ and $\int_{[0,1]} g^2 \, dm \leq 4$. Let $E = \{x \in [0,1] : f(x) + g(x) \geq 20\}$. Prove that $m(E) \leq 0.14$. (Hint: Consider the sets $\{x \in [0,1] : f(x) \geq 10\}$ and $\{x \in [0,1] : g(x) \geq 10\}$.)

4. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a measurable function. For each $L > 0$ define $f_L \colon \mathbb{R} \to \mathbb{R}$ as follows: $f_L(x) = f(x)$ for $x \in [-L, L]$ and $f_L(x) = 0$ for $x \notin [-L, L]$. Prove that if f is integrable over $\mathbb{R}$, then for each L, the function f_L is integrable over $\mathbb{R}$, and calculate the limit $\lim\limits_{L \to \infty} \int_{\mathbb{R}} f_L \, dm$.

5. (a) Give the definition of a function of bounded variation on $[0,1]$.
   (b) Construct a continuous function f on $[0,1]$ such that it has bounded variation, but its total variation on $[0,1]$ is not equal to $\int_0^1 |f'(t)| dt$. Support your answer.

6. Let $f \colon [0,1] \to \mathbb{R}$ be a function. Define $g(x,y) = f(x) - f(y)$ for $x, y \in [0,1]$.
   (a) Prove that if f is integrable on the interval $[0,1]$, then g is integrable on the square $[0,1] \times [0,1]$.
   (b) Prove that if g is integrable on the square $[0,1] \times [0,1]$, then f is integrable on the interval $[0,1]$.