

1. Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables which converges to 0 in probability as $n \rightarrow \infty$.

(a) Let $M > 0$ be a fixed number. Show that

$$\lim_{n \rightarrow \infty} E[|X_n| \mathbf{1}_{|X_n| \leq M}] = 0.$$

(b) Show that, if there exists constants $C > 0, \alpha > 1$ such that $E[|X_n|^\alpha] \leq C$ for all $n \geq 1$, then we have $\lim_{n \rightarrow \infty} E[|X_n|] = 0$.

2. Let $(Y_k)_{k=1}^{\infty}$ be a sequence of independent random variables with

$$P(Y_k = 1 + \sqrt{k}) = P(Y_k = 1 - \sqrt{k}) = \frac{1}{2}, k \in \mathbb{N}.$$

(a) Compute $E[Y_k]$ and $\text{Var}(Y_k)$.

(b) Show that there exists two sequences of constants $(a_n), (b_n)$ such that, as $n \rightarrow \infty$,

$$\frac{Y_1 + \dots + Y_n - a_n}{b_n}$$

converges in distribution to a standard normal random variable.

3. Let $(X_k)_{k=1}^{\infty}$ be i.i.d. random variables with probability density function

$$f(x) = \begin{cases} 2x^{-3} & \text{when } x \geq 1 \\ 0 & \text{when } x < 1. \end{cases}$$

(a) Prove that X_n/\sqrt{n} converges to 0 in probability as $n \rightarrow \infty$.

(b) Prove that X_n/\sqrt{n} does not converge to 0 almost surely as $n \rightarrow \infty$.

4. Let $(X_k)_{k=1}^{\infty}$ be the same i.i.d. sequence of random variables as in Problem 3. Let $M_n = \max_{k=1, \dots, n} X_k$ denote the maximum of X_1, \dots, X_n .

Prove that M_n/\sqrt{n} converges in distribution to a random variable Z , and identify the distribution of Z by finding its probability density function.