Preliminary Exam

Complex Analysis — January 2024

Please start each problem on a new page, and arrange your work in order when submitting.

Please choose 5 problems out of the 6 problems on the exam. Only 5 problems will be graded.

1. Suppose that f is an analytic function in a connected domain Ω satisfying $\text{Im}(f(z)) = (\text{Re}(f(z)))^2$ for all $z \in \Omega$. Show that f is constant on Ω.

2. Let $\Omega = \{z\colon |z| < 2\}$. Determine the number of zeros of the function $f(z) = z^5 - 3z^4 + \cos(z)$ in Ω.

3. Use the Residue Theorem to show that for $a > 1$

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos(\theta)} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

4. Let Ω be a simply connected bounded planar domain, and $f\colon \Omega \to \Omega$ be a complex analytic function such that $f(z_0) = z_0$ for some complex number $z_0 \in \Omega$. Suppose that for every complex analytic function $g\colon \Omega \to \Omega$ with $g(z_0) = z_0$ we have that $|g'(z_0)| \le |f'(z_0)|$. Prove that f is a bijective conformal map of Ω to itself.
   You may use the fact that every simply connected planar domain that is not the entire complex plane has a (bijective) conformal map from the unit disk to the domain so that it maps 0 *to* z_0.

5. Let $\Omega = \{z \in \mathbb{C}\colon \text{Re}(z) > 1\}$ and $f\colon \Omega \to \mathbb{C}$ be given by $f(z) = (z-1)^3$. Prove that there is an analytic function $g : \Omega \to \mathbb{C}$ such that for each $z \in \Omega$ we have $(g(z))^2 = f(z)$.

6. Find a conformal mapping f that maps the region S to T, where

$$S = T = \{z\colon \text{Im}(z) > 0\},$$

   satisfying

$$f(0) = 0, \qquad f(2) = 2, \qquad f(-2) = -1.$$

   Note that such a map extends to the Riemann sphere.