Preliminary Exam

Partial Differential Equations — January 2025

There are 6 problems (two pages) on this exam.
Please start each problem on a new page, and arrange your work in order when submitting.
Please choose 5 problems out of the 6 problems on the exam. Only 5 problems will be graded.

1. Suppose that $u\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic in $\mathbb{R}^n$ and u satisfies

$$\int_{\mathbb{R}^n} |Du(x)|^2 \, dx < \infty.$$

Prove that $u \equiv 0$ in $\mathbb{R}^n$.

2. Let $U \subset \mathbb{R}^n$ be open, bounded, and connected with C^1 boundary. Consider the *Robin* boundary value problem for Laplace's equation:

$$\begin{cases} \Delta u = u^3 & \text{in } U \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial U \end{cases}$$

where $\alpha > 0$ is a constant; and ν denotes the outward unit normal vector field on the boundary ∂U. Prove that this problem cannot have a smooth solution $u\colon \overline{U} \to \mathbb{R}$ in $C^2(U) \cap C(\overline{U})$ other than $u \equiv 0$.

3. Let $\Omega \in \mathbb{R}^2$ be an open, bounded, and connected domain with smooth boundary $\partial\Omega$. Show that the nonlinear boundary-valued problem

$$\begin{cases} \Delta u(x,y) + y u_x(x,y) - 2x u_y(x,y) - u(x,y)^5 = 0, & (x,y) \in \Omega, \\ u(x,y) = 0, & (x,y) \in \partial\Omega, \end{cases}$$

has no non-trival solutions.

4. Let $L > 0$ and $T > 0$ be given. Assume that $u(x,t)$ satisfies

$$\begin{cases} u_t - u_{xx} + h(x,t)u \geq 0, & 0 < x < L, \quad 0 < t < T, \\ u(x,0) \geq 0, & 0 < x < L, \\ u(0,t) \geq 0, & 0 < t < T, \\ u(L,t) \geq 0, & 0 < t < T, \end{cases}$$

where $h(x,t)$ is any function satisfying $|h(x,t)| \leq M$ for all $0 < x < L$ and $0 < t < T$, and some constant $M > 0$. Show that

$$u(x,t) \geq 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t < T.$$

Hint: First suppose that $0 < h(x,t) \leq M$ and show that the claim follows. Next, reduce the general case to the case you have just proved using this hint.

5. Consider the initial-value problem

$$\begin{cases} u_t + u_x = \dfrac{1}{3} u^4, & x \in (-\infty, \infty), \quad t > 0, \\ u(x,0) = (1 + x^2)^{-\frac{1}{3}}, & x \in (-\infty, \infty), \quad t = 0. \end{cases}$$

(a) Solve the initial-value problem.

(b) Does the solution blow up in finite time?

(c) If the solution blows up in finite time, what is the earliest time it blows up?

6. Compute explicitly the unique entropy solution of

$$\begin{cases} u_t + uu_x = 0, & x \in (-\infty, \infty), \quad t > 0, \\ u(x,0) = g(x), & x \in (-\infty, \infty), \quad t = 0, \end{cases}$$

with

$$g(\zeta) = \begin{cases} 0 & \text{if } \zeta < -1, \\ 2 & \text{if } -1 < \zeta < 1, \\ 1 & \text{if } 1 < \zeta. \end{cases}$$