Real Analysis Preliminary Exam, January 2025

Time allowed: 2 hours 30 minutes.

This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted.

Notation: $\mathbb{R}$ is the set of real numbers; m is the Lebesgue measure on $\mathbb{R}$; m^* is the Lebesgue outer measure on $\mathbb{R}$.

1. Let $\{A_n\}_n$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $A_{n+1} \subset A_n$ for each n. Let $A \subset \mathbb{R}$ be a set such that $A \subset A_n$ for all n and $\lim_{n\to\infty} m^*(A_n \setminus A) = 0$. Prove that A is Lebesgue measurable.

2. Let f be a Lebesgue integrable real-valued function on $\mathbb{R}$. Show that for any $\varepsilon > 0$ there exists a bounded interval I such that $\displaystyle\int_{\mathbb{R}\setminus I} |f|\, dm \le \varepsilon$.

3. Let $\{f_n\}_n$ be a sequence of Lebesgue measurable functions that converges a. e. on $\mathbb{R}$. Assume that there exists a subsequence $\{n_j\}$ such that
$$\lim_{j\to\infty} \int_{\mathbb{R}} |f_{n_j}|\, dm = 0.$$
Show that f_n converges to 0 a. e. on $\mathbb{R}$.

4. (a) Give the definition of an absolutely continuous function on $I = [0, 1]$.
   (b) Let f be an absolutely continuous function on I and let $E \subset I$ be such that $m(E) = 0$. Show that $m(f(E)) = 0$.

5. Let f be a non-negative Lebesgue measurable function on $[0, 1]$. Prove that f is Lebesgue integrable if and only if
$$\sum_{n=1}^{\infty} m\big(\{x \in [0,1] \colon f(x) > n\}\big) < \infty.$$
(Hint: the Integral Test for series may be useful.)

6. Let $(X, \mathcal{M}, \mu_1)$ be a *finite* measure space, and f be a non-negative integrable function on this space. Let μ_2 be a measure on $(X, \mathcal{M})$ such that the Radon–Nikodym derivative $\frac{d\mu_2}{d\mu_1}$ is equal to f. Let g be a non-negative integrable function on $(X, \mathcal{M}, \mu_2)$ and for all $A \in \mathcal{M}$ let $\mu_3(A) = \int_A g\, d\mu_2 + 2\mu_2(A) + 3\mu_1(A)$.
   (a) Prove that μ_3 is a finite measure on $(X, \mathcal{M})$.
   (b) Prove that $\mu_3 \ll \mu_1$ and find the Radon–Nikodym derivative $\frac{d\mu_3}{d\mu_1}$.