**This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted. (An incorrect problem was deleted; best four out of five solutions were counted.) Proofs or counterexamples are required for all problems. Time allowed: 2 hours 30 minutes.**

$\mathbb{R}$ **denotes the real line;** $\mathbb{R}^d$ **is the** $d$**-dimensional Euclidean space with the usual norm** $\|x\| = \left(\sum_{k=1}^{d} x_k^2\right)^{1/2}$.

1. Suppose that $f_1$ and $f_2$ are two differentiable functions on the interval $[-1,1]$ such that $f_1(x) \le f_2(x)$ for all $x \in [-1,1]$, $f_1(0) = f_2(0)$, and $f_1'(0) = f_2'(0)$. Show that if a function $f : [-1,1] \to \mathbb{R}$ is such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [-1,1]$, then $f$ is differentiable at $0$.

2. This problem was deleted

3. (a) Give the definition of what it means for a sequence $\{a_n\}$ of real numbers to be a Cauchy sequence.

   (b) Let $\{x_n\}$ be a sequence defined as follows:

$$x_1 = 1; \qquad x_{n+1} = x_n + \frac{\cos(x_n)}{n^2},\ n \ge 1.$$

   Prove that $\{x_n\}$ is a Cauchy sequence.

4. In this question, $I$ is an interval in the real line.

   (a) Give the definition of what it means for a function $f : I \to \mathbb{R}$ to be uniformly continuous on $I$.

   (b) Prove that the function $f\colon [1,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous.

5. Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Let $f_+$ be the function defined by $f_+(x) = \max\{f(x), 0\}$.

   (a) Suppose that $f$ is Riemann integrable. Is $f_+$ a Riemann integrable function? Prove or give a counterexample.

   (b) Suppose that $f_+$ is Riemann integrable. Is $f$ a Riemann integrable function? Prove or give a counterexample.

6. (a) Give the definition of what it means for a sequence of functions $\{f_n\}$ on an interval $I$ to converge uniformly on $I$ to a function $f$.

   (b) Let $f_n(x) = (\sin(x))^n$ for $n \ge 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0, \pi/2]$.

   (c) In part (b), does the sequence $\{f_n\}$ converge to $f$ uniformly on $[0, \pi/2]$? Support your answer.