Linear algebra qualification exam

May 3, 2024

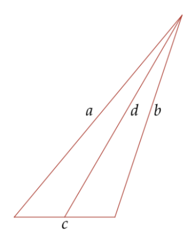
1. Let V be a finite-dimensional vector space over $\mathbb{R}$, and $T : V \to V$ be linear. Suppose $T^3 v = v$ for all $v \in V$, and $T^2 v \neq v$ for all nonzero $v \in V$.

   a) Show that T has no real eigenvalues.

   b) Show that $\dim V$ cannot be odd.

2. Prove Apollonius's identity, using properties of the inner product: in a triangle in $\mathbb{R}^n$ with sides of length a, b, and c, let d be the line segment from the midpoint of the side length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$



3. Let V and W be vector spaces, with $S, T : V \to W$ linear. Let $U = \{v \in V : Sv = Tv\}$.

   a) Show that U is a subspace of V.

   b) Suppose S is injective. Show that $\dim U \leq \dim \text{range}\,(T)$.

4. Let V be a finite-dimensional vector space, and let $S = \{v_1, \ldots, v_k\} \subseteq V$. Prove the following.

   a) If S is linearly independent, then S can be completed to a basis of V.

   b) If S spans V, then S contains a basis of V.

5. Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively, and write $\mathcal{L}(V, W)$ for the set of all linear maps from V to W. Prove that $\mathcal{L}(V, W)$ is a vector space. What is its dimension?