Ordinary Differential Equations Preliminary Exam
April 2024

Answer 4 of the 5 questions.

1. Prove the following properties of the matrix exponential for $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{N\times N}$.
   (a) If $\mathbf{XY} = \mathbf{YX}$, then $e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{X}+\mathbf{Y}}$.
   (b) If $\mathbf{X}$ is anti-symmetric, then $e^{\mathbf{X}}$ is an orthogonal matrix.

2. Determine and classify the steady states for the ODE

$$\frac{dx^2}{dt^2} = x\left(1 - x^2\right)$$

   and sketch the phase plane.

3. Consider the following ODE:

$$y'' + f(y)\, y' + y = 0,$$

   where $y = y(t)$.
   (a) Convert the equation into a system by letting $x_1 = y$ and $x_2 = y'$.
   (b) Show that the rest point $(0, 0)$ of this system is asymptotically stable, provided that $f(y) > 0$ for all y.
   (c) What does this imply for the original equation?

4. Consider the initial value problem

$$\frac{\partial \boldsymbol{x}}{dt} = \mathbf{A}\boldsymbol{x}$$

   with the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   (a) Find the eigenvalues of $\mathbf{A}$.
   (b) Solve the initial value problem.

5. Consider the nonlinear system

$$\begin{aligned} \frac{\partial x}{\partial t} &= -2y + yz - x^3 \\ \frac{\partial y}{\partial t} &= x - xz - y^3 \\ \frac{\partial z}{\partial t} &= xy - z^3 \end{aligned}$$

   (a) Find its linearized system at the equilibrium point $(0, 0, 0)$.
   (b) Is the linearized system obtained in (a) asymptotically stable?
   (c) Show that the nonlinear system is asymptotically stable.