Real Analysis Preliminary Exam, May 2024

Time allowed: 2 hours 30 minutes.

This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted.

Notation: $\mathbb{R}$ is the set of real numbers; m is the Lebesgue measure on $\mathbb{R}$; m^* is the Lebesgue outer measure on $\mathbb{R}$.

1. Let $A \subset \mathbb{R}$ and $B = \{x \in \mathbb{R} \colon 3x \in A\}$. Prove that $m^*(A) = 3m^*(B)$.

2. Let f be a non-negative measurable function on $[0, 1]$ such that $\int_{[0,1]} f^2\, dm < 1$. Let $E = \{x \in [0, 1] : f(x) < 10\}$. Show that $m(E) \geq \frac{99}{100}$.

3. Let $\{f_n\}_n$ be a sequence of non-negative measurable functions that converges to an integrable function f a.e. on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} \frac{f}{1 + (f_n - f)^2}\, dm = \int_{[0,1]} f\, dm.$$

4. (a) Give the definition of an absolutely continuous function on $[0, 1]$.
   (b) Let f be an absolutely continuous function on $[0, 1]$. Prove that the function $\sin(f)$ is also absolutely continuous on $[0, 1]$.

5. Let f and g be two non-negative integrable functions on $\mathbb{R}$; let f be continuous. Define

$$h(x) = \int_{\mathbb{R}} f(x - y)g(y)\, dm(y) \qquad \text{for } x \in \mathbb{R}.$$

   Prove that h is measurable and $\int_{\mathbb{R}} h(x)\, dm(x) = \int_{\mathbb{R}} f\, dm \int_{\mathbb{R}} g\, dm$.
   (If you use a theorem, don't forget to justify its assumptions)

6. Let $(X, \mathcal{M}, \mu)$ be a *finite* measure space, and f be a non-negative measurable function on this space.
   (a) Prove that if f^2 is integrable on $(X, \mathcal{M}, \mu)$, then so is f.
   (b) Under the assumptions of part (a), define the following two finite measures on $(X, \mathcal{M})$:

$$\nu_1(E) = \int_E f\, d\mu,\ E \in \mathcal{M}; \quad \nu_2(E) = \int_E f^2\, d\mu,\ E \in \mathcal{M}.$$

   Prove that $\nu_2 \ll \nu_1$ and find the Radon–Nikodym derivative $\frac{d\nu_2}{d\nu_1}$.