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PH.D. QUALIFYING EXAM
APRIL 2013

Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) (a) State what it means for a function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ to be *continuous at* $(a, b) \in \mathbb{R}^2$.
(b) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} x^2 + y^2 & \text{if } x, y \in \mathbb{Q}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Prove that f is continuous at $(0, 0)$ and that f is not continuous at any point $(a, b) \neq (0, 0)$.

(2) Let $a_0, a_1, \cdots, a_n$ be real numbers with property that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots \frac{a_n}{n+1} = 0.$$

Prove that the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n = 0$$

has at least one solution in $(0, 1)$. (Hint: use Rolle's theorem.)

(3) For each $n \in \mathbb{N}$, define $f_n(x) := e^{-nx^2}$. Show that for any given $0 < a < 2$, $(f_n)_{n=1}^{\infty}$ converges uniformly on $[a, 2]$ and

$$\lim_{n\to\infty} \int_a^2 f_n(x)\, dx = 0\,.$$

(4) Let $W := \{(a, b, c, d, e) \in \mathbb{R}^5 \mid a = 3c,\ 2b = 4d - 5e\}$.
(a) Show that W is a vector subspace of $\mathbb{R}^5$.
(b) Find bases for W and for $W^{\perp} := \{v \in \mathbb{R}^5 \mid v \perp w \text{ for all } w \in W\}$.

(5) Suppose that a 2×2 matrix A has eigenvalues 1 and 0. Prove that $A^2 = A$.

(6) Let $(a_n)_1^{\infty}$ and $(b_n)_1^{\infty}$ be sequences of positive real numbers. Assume that the limit

$$L := \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is positive and finite. Prove that $\sum_1^{\infty} a_n$ and $\sum_1^{\infty} b_n$ either both converge or both diverge.

(7) Define $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2$ by $F(x, y) := (e^x - e^y, e^x + e^y)$. Prove that F is locally invertible at each point of $\mathbb{R}^2$. State explicitly what this means for the point $(x, y) = (1, 1)$, and find the derivative of the inverse of F at the point $F(1, 1)$.

Date: April 11, 2013.