PHD PRELIMINARY EXAM IN ALGEBRA AND TOPOLOGY
2017-05 May 2017

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI

Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

I. Algebra

(1) Let $\alpha = \sqrt{2+\sqrt{2}}$.
  (a) Find the minimum polynomial $p(x)$ over $\mathbb{Q}$ that α satisfies.
  (b) Find the splitting field E of $p(x)$ over $\mathbb{Q}$
  (c) Find the Galois group, $\mathrm{Gal}(E/\mathbb{Q})$.

(2) Let $F \subset K \subset L$ be a tower of field extension.
  (a) If $[L:K] = m < \infty$ and $[K:F] = n < \infty$, prove that $[L:F] = mn$.
  (b) If L is algebraic over K and K is algebraic over F, show that L is algebraic over F.

(3) Let F be a splitting field over $\mathbb{Q}$ of the polynomial $f(x) = x^5 - 3 \in \mathbb{Q}[x]$. Use the Fundamental Theorem of Galois Theory to show that the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ of $f(x)$ over $\mathbb{Q}$ contains a subgroup of order 4 that is not normal. Conclude that $\mathrm{Gal}(F/\mathbb{Q})$ is not Abelian.

(4) (a) Prove that every irreducible polynomial over a finite field F is separable and a polynomial in $F[x]$ is separable if and only if it is the product of distinct irreducible polynomials.
  (b) Do inseparable extension exist? If yes, please give an example; if no, prove it.

II. Topology.

1. Let X and Y be topological spaces; let $A \subset X$ and $B \subset Y$. Assume that $f : X \to Y$ is a continuous map. Proof or give a counter-example:
   (1) if A is closed, then $f(A)$ is closed;
   (2) if B is closed, then $f^{-1}(B)$ is closed;
   (3) if A is connected, then $f(A)$ is connected;
   (4) if B is connected, then $f^{-1}(B)$ is connected.

2. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1.
   (1) Show that q is a quotient map.
   (2) Use examples to show that q is neither open nor closed.

3. Recall that a space is *countably compact* if every countable cover has a finite subcover. Prove that if $f : X \to \mathbf{R}$ is a continuous, real valued function defined on the countably compact space X, then f is bounded.

4. Give examples of path-connected topological spaces such that their fundamental groups are, respectively, $\mathbb{Z}_n$ and $\mathbb{Z} \times \mathbb{Z}_n$. Justify your answers.