PhD Preliminary Exam in Algebra and Topology

April 29, 2013

Department of Mathematical Sciences

University of Cincinnati

Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

Algebra

(1) Let $\mathbb{Q}$ be the field of rational numbers and let $\mathbb{R}$ be the field of real numbers. Let $\zeta = e^{2\pi i/13}$, a complex primitive 13-th root of unity. Prove that $\mathbb{Q}(\zeta)$ contains exactly one subfield K such that $\dim_{\mathbb{Q}} K = 6$. Prove further that K is a Galois extension of $\mathbb{Q}$ and that $K \subset \mathbb{R}$.

(2) An ideal I in a commutative ring R with unit is called primary if $I \neq R$ and whenever $ab \in I$ and $a \notin I$, then $b^n \in I$ for some positive integer n. Prove that if R is a principal ideal domain, then I is primary if and only if $I = P^n$ for some prime ideal P of R and some positive integer n.

(3) Let $F \subset E$ be an extension of fields such that $\dim_F E$ is finite. Define what is meant for such an extension to be a) normal; and b) separable. Let p be a prime and let $\mathbb{F}_p$ be the field with p elements; Let $E = \mathbb{F}_p(t)$, the field of rational functions in the indeterminate t and let $F = \mathbb{F}_p(t^p)$ be the subfield generated by t^p. Prove that the extension $E \supset F$ is normal but not separable.

(4) Let F be a finite field and let F^* denote the multiplicative group of non-zero elements. Prove that F^* is cyclic. Deduce that any extension of finite fields is simple. Prove that if $|F| = q$, then

$$X^q - X = \prod_{\alpha \in F} (X - \alpha)$$

Deduce that any finite extension of fields is normal and separable.

Topology

(1) Let $\mathbb{R}_K$ denote the real line with K-topology generated by the collection of all open intervals (a, b) along with all sets of the form $(a, b) \setminus K$, where K is the set of all numbers of the form $\frac{1}{n}$, n is a positive integer. Let Y the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point; let $p : \mathbb{R}_K \to Y$ be the quotient map. Prove the following statements.
   (a) Y is a connected space.
   (b) Y is not a Hausdorff space.
   (c) $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

(2) Prove or disprove: Any continuous map $f : \mathbb{RP}^2 \to \mathbb{T}$ from the real projective plane to a torus is homotopic to a constant map.

(3) (a) Let X be the space obtained by gluing the boundary of the closed unit disk D^2 to the unit circle S^1 by the map $z \to z^n$, where n is a positive integer. Find the fundamental group of X.
(b) Find a space whose fundamental group is $\mathbb{Z}_3 \times \mathbb{Z}_5$.
(Justify your answers.)

(4) Let $\mathbb{RP}^2$ be the real projective plane; let X be the one point union $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
(a) Compute $\pi_1(X)$.
(b) Find the universal covering space of X. (A description of a covering space includes both a definition of the space as well as the definition of the covering map, and an indication of why the map is a covering map.)