PhD Preliminary Exam in Algebra and Topology
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Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

Algebra

(1) (a) Prove that prime elements in an integral domain are irreducible.
(b) Let D be a principal ideal domain. Prove that if P is a non-zero prime ideal in D, then P is a maximal ideal.
(c) Let $R[x]$ be the ring of polynomials in one indeterminate over an integral domain R. Prove that if $R[x]$ is a principal ideal domain, then R is a field.

(2) Let ζ be a primitive 10th root of unity in $\mathbb{C}$.
(a) Factor the polynomial $x^{10} - 1$ into irreducibles and identify the minimum polynomial of ζ among the irreducible factors.
(b) Find the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
(c) Find all intermediate fields between $\mathbb{Q}(\zeta)$ and $\mathbb{Q}$.

(3) Let F be a finite field with $|F| = q$. Let $G : F \to F$ be the map $G(x) = x^n$. Find necessary and sufficient conditions for G to be invertible and find the inverse of G in the case when it is invertible.

(4) Let G be a group and K a field. A character of G in K is a homomorphism from G to the multiplicative group of the field, $\chi : G \to K^*$. Functions $f_1, \dots, f_n : G \to K$ are said to be linearly independent if for $a_1, \dots a_n \in K$, $a_1 f_1 + \cdots + a_n f_n = 0$ implies $a_1 = \cdots = a_n = 0$. Show that if $\chi_1 \dots, \chi_n$ are distinct characters of G in K, then they are linearly independent.

Topology

(1) Let $A = \mathbb{Q} \cap (0, 1)$ be a subspace of $\mathbb{R}$. Identify the closure of A with respect to these topologies:
   (a) The lower-limit topology $\mathcal{T}_l$ generated by intervals of the form $[a, b)$.
   (b) The topology $\mathcal{T}_+$ generated by intervals of the form $[a, +\infty)$.
   (c) The finite-complement topology $\mathcal{T}_f$.
   (d) The countable-complement topology $\mathcal{T}_c$.

   For each of the four parts, justify your answer.

(2) Given topologies $\mathcal{T}, \mathcal{T}'$ on a space X, let $\mathcal{T} \vee \mathcal{T}'$ denote the topology generated by the subbasis $\mathcal{T} \cup \mathcal{T}'$. Find an example of topologies $\mathcal{T}, \mathcal{T}'$ on the real line such that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}')$ are both compact, but $(\mathbb{R}, \mathcal{T} \vee \mathcal{T}')$ is not.

(3) Show that any $g : S^1 \to S^1$ that is homotopic to the identity map must be surjective.

(4) Let $p : E \to X$ be a covering map and suppose that continuous maps $f, g : Y \to E$ satisfy $p \circ f = p \circ g$ (in other words, f and g are both liftings of the same map $h : Y \to X$). Show that if Y is path-connected and f agrees with g at some point $y_0 \in Y$, then $f(y) = g(y)$ at every $y \in Y$.