PHD PRELIMINARY EXAM IN ALGEBRA AND TOPOLOGY 2016

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI

Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

I. Algebra

$\mathbb{Z}$ is the integer ring and $\mathbb{Q}$ is the rational number ring (field).

1. a) Prove that every Euclidean domain is a Principal Ideal Domain.
b)If R is a PID and J is a nonzero ideal of R, show that J is maximal if and only if J is prime.

2. Let I be the ideal generated by 3 and $x^3 - x + 1$ in the ring $\mathbb{Z}[x]$. Is I a prime ideal? Is I a maximum ideal? Prove your claims. Give an explicit description of the quotient ring R/I

3. Let $f(x) = x^9 + x^8 + \cdots + x^2 + x + 1$. Let K be the splitting field for $f(x)$ over $\mathbb{Q}$. Determine $[K : \mathbb{Q}]$ and find $\mathrm{Gal}(K/\mathbb{Q})$. Prove that there is a unique quadratic extension $\mathbb{Q}[\sqrt{d}]$ of $\mathbb{Q}$ inside K, and determine d.

4. a) Let $F \subset K$ be fields. If $a, b \in K$ are algebraic over F, prove that both $a + b$ and ab are algebraic over F.
b) Let K be a finite algebraic extension of F. Show that $K = F(\alpha)$ for some $\alpha \in K$, that is, show that K is a simple extension of F.

II. Topology

1. Given a topological space X, for $x, y \in X$ we say that $x \sim y$ if every open subset of X that contains x also contains y, and every open subset of X that contains y also contains x.
(a) Prove that $\sim$ is an equivalence relation on X.
(b) Let $Y = X/\sim$ be equipped with the quotient topology from X, and let $[x], [y] \in Y$ such that $[x] \neq [y]$. Show that there is an open (in the quotient space topology) set $W \subset Y$ such that either $[x] \in W$ and $[y] \notin W$, or $[x] \notin W$ and $[y] \in W$. (This means that the quotient space topology is T_1.)

2. Let $X \subset R^2$ be the union of lines $\{1/n\} \times R$ and $R \times \{1/n\}$, and the point (0,0). Is X path-connected?

3. Prove that if X is a simply connected space and (E, p) is a covering space of X, then p is a homeomorphism of E onto X.

4. Let S^1 be the unit circle and D the closed unit disk in $\mathbb{R}^2$, equipped with the Euclidean topology. Fix a positive integer $n > 1$ and let $f : S^1 \to D$ be given by $f(e^{i\theta}) = e^{in\theta}$. Compute the fundamental group of $D \cup_f D$, where $D \cup_f D$ is the topological space obtained by gluing one copy of D along its boundary S^1 to the other copy of D using the gluing map f.