ANALYSIS PRELIMINARY EXAMINATION, SPRING 2015.

Part I: REAL ANALYSIS

Here, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^2$ the 2-dimensional plane, and m denotes the Lebesgue measure on $\mathbb{R}$.

1. Answer the following two parts.

(a) Is $\mathbb{R}^2$ a countable union of circles? Prove or disprove.

(b) Is there a measurable set $E \subset [0,1]$ for which $m(E \cap [0,x]) = x^2$ for almost every $x \in [0,1]$? Prove or disprove.

2. Let $f, g \in L^1(\mathbb{R}, m)$ be both non-negative. Show that $h \in L^1(\mathbb{R}, m)$, where for $x \in \mathbb{R}$, the value $h(x)$ is given by

$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

Hint: First show the measurability of the map $(x,t) \mapsto g(x-t)$ by looking at this map as a composition of two functions, $\mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{\sin(x)}{x}$ when $x \neq 0$ and $f(0) = 1$. Show that f is *not* Lebesgue integrable on $[0, \infty)$.

4. Show that the series $\sum_{n=1}^{\infty} x^n e^{-nx}$ converges for $x > 0$, and that

$$\int_0^{\infty} \left[\sum_{n=1}^{\infty} x^n e^{-nx} \right] dx = \sum_{n=1}^{\infty} \int_0^{\infty} x^n e^{-nx}\, dx.$$

Part II: COMPLEX ANALYSIS

Here $\mathbb{C}$ is the complex plane and $\mathbb{D}$ denotes the unit disk centered at 0 and $\overline{\mathbb{D}}$ is the closed unit disk centered at 0.

1. Let f be analytic on a domain Ω that contains $\overline{\mathbb{D}}$.

(a) Show that

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

(b) Use part (a) to show that whenever $z \in \mathbb{D}$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{e^{i\theta} + z}{1 + \overline{z}\, e^{i\theta}} \right) d\theta.$$

Hint: Consider conformal self-maps of the unit disk.

2. Find a power series expansion of

$$f(z) = \frac{3z - 5}{z^2 - 4z + 3}$$

valid for $z = 2$. In what domain is your expansion valid?

3. Use complex variable techniques to compute

$$\int_0^{\infty} \frac{1}{x^4 + 16} dx.$$

Give justification for all your steps.

4. Find all the conformal mappings from Ω to the unit disk, where

$$\Omega = \{z \in \mathbb{C} : |z| > 1, \mathrm{Re}(z) > 0, \text{ and } \mathrm{Im}(z) > 0\},$$

such that the image of $1 + i$ is 0.