

**REAL & COMPLEX ANALYSIS PRELIMINARY EXAM, FALL 2015,  
UNIVERSITY OF CINCINNATI**

**Real Analysis:** In this portion of the exam,  $\mathbb{R}$  denotes the field of real numbers and  $m$  denotes the Lebesgue measure on  $\mathbb{R}$ . Statements regarding measurability and almost everywhere are with respect to the Lebesgue measure.

1. State the fundamental theorem of calculus for absolutely continuous functions, and use it to answer the following question: is there a measurable set  $E \subset [0, 1]$  such that  $m([0, x] \cap E) = x/3$  for almost every  $x \in [0, 1]$ ?

2. Let  $f$  be an integrable function on  $\mathbb{R}$  such that for every continuous function  $g$  on  $\mathbb{R}$ ,  $\int_{\mathbb{R}} f(x)g(x) dm(x) = 0$ . Prove that  $f$  is zero almost everywhere on  $\mathbb{R}$ .

3. A function  $g : [a, b] \rightarrow \mathbb{R}$  is *Lipschitz* if there is a constant  $M \geq 0$  such that  $|g(x) - g(y)| \leq M|x - y|$  for all  $x, y \in [a, b]$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous. Prove that  $f$  is Lipschitz if and only if there exists  $L \geq 0$  and a set  $E \subset [a, b]$  such that  $m(E) = 0$  such that  $f$  is differentiable at each  $x \in [a, b] \setminus E$  with  $|f'(x)| \leq L$ .

4. Let  $f$  be a Lebesgue measurable function on  $[0, 1]$  with  $f(x) > 0$  almost everywhere. Suppose that  $\{E_k\}_k$  is a sequence of Lebesgue measurable sets in  $[0, 1]$  such that

$$\lim_{k \rightarrow \infty} \int_{E_k} f(x) dx = 0.$$

Show that  $\lim_{k \rightarrow \infty} m(E_k) = 0$ .

**Complex Analysis:** In this portion of the exam,  $\mathbb{C}$  denotes the complex plane, and the real part and the imaginary part of a complex number are denoted  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$ .

1. Let  $\Omega$  be a planar domain and  $U \subset \Omega$  be a non-empty open set. For each positive integer  $n$  let  $f_n : \Omega \rightarrow \mathbb{C}$  be analytic. Suppose that this sequence of functions converges uniformly on  $U$  to a function  $f : U \rightarrow \mathbb{C}$ . Show that  $f$  is analytic on  $U$ .

2. Let  $\Omega$  be a planar domain, and  $z_0 \in \Omega$ ,  $r > 0$  such that the open disk  $D(z_0, r) \subset \Omega$ . Characterize the set of analytic functions  $f : \Omega \rightarrow \mathbb{C}$  such that  $\operatorname{Re}(f(z)) = (\operatorname{Im}(f(z)))^2$  for  $z \in D(z_0, r)$ .

3. For each real number  $a > 1$  compute the following integral:

$$\int_0^{2\pi} \frac{1}{a + \cos \theta} d\theta.$$

4. Find all the conformal maps from the domain  $\Omega_1 = \{z \in \mathbb{C} : |z|^2 < 1 \text{ and } |z - 1/2|^2 < 1/4\}$  to  $\Omega_2 = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re}(z) > 0, \operatorname{Im}(z) = 0\}$  so that the part of the real axis in the domain  $\Omega_1$  is mapped to the part of the unit circle centered at 0, lying in  $\Omega_2$ .