Differential Equations Preliminary Exam
April 2018

I. Ordinary Differential Equations

1. Consider $\dot{x} = Ax$, where A is a constant $n \times n$ matrix.
   (a) Define e^{At} and show that the solution of the IVP $x(0) = x_0$ is $x(t) = e^{At}x_0$.
   (b) Show that $e^{A(t+\tau)} = e^{At}e^{A\tau}$ by using a uniqueness theorem for solutions of the system.
   (c) Find the solution of $\dot{x} = Ax + f(t)$ in terms of e^{At} and $f(t)$.

2. Consider the nonlinear system

$$x' = x - y, \quad y' = x^2 + y^2 - 2.$$

   (a) Find the fixed points and classify them by computing the linearization about each.
   (b) Carefully sketch the phase portrait.

3. Consider the damped harmonic oscillator

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx - \epsilon y^3(1 + x^2) \end{aligned}$$

   where x represents the displacement of the spring and k is the spring constant with $k > 0$.
   (a) Explain why linear analysis at the origin is not useful to determine the stability of the origin.
   (b) Use an appropriate Liapunov function to determine the nature of the origin.

4. Consider the system

$$\begin{aligned} \dot{x} &= -y + x(1 - 2x^2 - 3y^2) \\ \dot{y} &= x + y(1 - 2x^2 - 3y^2) \end{aligned}$$

   (a) Find equilibrium points and determine their stability.
   (b) Rewrite the equations in polar coordinates.
   (c) Use the Poincare-Bendixson theorem to show that there is a closed orbit in the annulus $A = \{(x, y) : r_0 \leq \sqrt{x^2 + y^2} \leq R\}$.

II. Partial Differential Equations

1. Solve for $u(x,y)$ using characteristics:

$$xu_x + yu_y = 2u, \quad u(x,1) = g(x).$$

2. Let $g(x) \geq 0$ be a non-negative function on the circle $\{|x| = 2\} \subset \mathbb{R}^2$. Prove that there is no smooth solution of

$$\begin{cases} -\Delta u = 0 & \text{for all } \ |x| < 2, \\ u(x) = g(x) & \text{for all } \ |x| = 2 \end{cases}$$

with the values $u(0,0) = 1$ and $u(1,0) = 4$.

3. Let $U \subset \mathbb{R}^n$ be bounded, and let $u(x,t)$ be a smooth solution of

$$\begin{cases} u_{tt} - \Delta u + u^3 = 0 & \text{in } \ U \times (0,T] \\ u(x,t) = 0 & \text{on } \ \partial U \times [0,T] \\ u(x,0) = 0 & \text{in } \ U \\ u_t(x,0) = g(x) & \text{in } \ U \end{cases}$$

Show that at every $t > 0$, $\displaystyle\int_U \tfrac{1}{2}[u_t(x,t)]^2 + \tfrac{1}{2}|Du(x,t)|^2 \, dx \leq \int_U \tfrac{1}{2}[g(x)]^2 \, dx.$

4. Let $\phi(r)$ be a solution of the ODE $\phi^{(4)}(r) - \frac{1}{4}r\phi'(r) - \frac{1}{4}\phi(r) = 0$ with the additional property that $\displaystyle\int_{-\infty}^{\infty} \phi(r)\, dr = 1.$

a) Show that $\Phi(x,t) = \dfrac{1}{\sqrt[4]{t}}\phi\Big(\dfrac{x}{\sqrt[4]{t}}\Big)$ is a solution of $\Phi_t + \Phi_{xxxx} = 0$ in the domain $\mathbb{R} \times (0,\infty)$.

b) Suggest a formula for solving the initial-value problem

$$\begin{cases} u_t + u_{xxxx} = 0 \ \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x). \end{cases}$$

c) Suggest a formula for solving the inhomogeneous initial-value problem

$$\begin{cases} u_t + u_{xxxx} = f(x,t) \ \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 0. \end{cases}$$