Preliminary Exam

Differential Equations

August 22, 2013

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ——————- Problem 2.1——————

Problem 1.2 ——————- Problem 2.2——————

Problem 1.3——————- Problem 2.3——————

Problem 1.4——————- Problem 2.4——————

Part I total score : —————- Part II total score ——————————

Total score ———————

Part I: Ordinary Differential Equations

Problem 1.1: Suppose A is an $n \times n$ matrix and $t \in \mathbb{R}$.

1. Define the matrix exponential e^{tA}.
2. Show that $\frac{d}{dt}e^{At} = Ae^{At}$.

Problem 1.2: Each of the following systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has an equilibrium at $(0, 0)$. For each system, determine which of the following behaviors are possible:

saddle • unstable node • stable node • unstable focus • stable focus • center • center focus

Explain your reasoning. Be sure to note any properties of $\mathbf{f}(\mathbf{x})$ you use.

1. $$\begin{aligned}\dot{x} &= x - 3y + y^5 \\ \dot{y} &= 4x - y + xy^4 + x^2y^2\end{aligned}$$

2. $$\begin{aligned}\dot{x} &= 5x + 4\sin y + 2x^2y - xy^3 \\ \dot{y} &= x + 2y + x^3 + 7xy\end{aligned}$$

Problem 1.3: State and prove Gronwall's inequality.

Problem 1.4: Let $\mathbf{f}$ be a C^1 vector field in an open set $E \subset \mathbb{R}^2$ containing an annular region A with a smooth boundary. Suppose that $\mathbf{f}$ has no zeros in the closure of A, and that $\mathbf{f}$ is transverse to the boundary of A, pointing inward. Show that A contains a periodic orbit. Also show that if A contains a finite number of cycles $\{C_1, \ldots, C_m\}$, then A contains at least one stable limit cycle.

Part II: Partial Differential Equations

Problem 2.1 Let u be a smooth function satisfying

$$\Delta u = 0$$

and $u \geq 0$ on the upper half-plane $\{(x_1, x_2) : x_2 > 0\}$. Suppose $u(0,2) = 1$. Prove that $u(0,3) \leq 4$.

Problem 2.2 Let $U \subset \mathbf{R}^n$ be a bounded domain with smooth boundary, and u a solution of the heat equation with boundary conditions

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0,T] \\ \dfrac{\partial u}{\partial \nu} = -u & \text{on } \partial U \times (0,\infty) \\ u(x,0) = g(x) & \text{on } U \times \{t=0\} \end{cases}.$$

Suppose that $g(0) > 0$. Show that $\max\limits_{\overline{U}_T} u(x,t) = \max\limits_{\overline{U}} g(x)$.

Problem 2.3 Let $U \subset \mathbf{R}^n$ be open and bounded, with smooth boundary, and $c(x,t)$ a nonnegative function on $U \times (0,\infty)$. Show that a smooth solution to the PDE

$$\begin{cases} u_{tt} + c(x,t)u_t - \Delta u = 0 & \text{in } U \times (0,T] \\ u(x,t) = 0 & \text{on } \partial U \times [0,T] \\ u(x,0) = 0 & \text{on } U \times \{t=0\} \\ u_t(x,0) = h(x) & \text{on } U \times \{t=0\} \end{cases}$$

satisfies the inequality $\displaystyle\int_U u_t^2 + |Du|^2 \, dx \leq \int_U h^2 \, dx$ at every $t > 0$.

Problem 2.4-a Find the entropy solution to the equation $\begin{cases} u_t + e^u u_x = 0 & \text{in } \mathbf{R} \times (0, \infty) \\ u(x,0) = & \begin{cases} 1 \text{ if } x < 0 \\ 0 \text{ if } x \geq 0 \end{cases} \end{cases}$.

Problem 2.4-b. Find the entropy solution to the equation $\begin{cases} u_t + e^u u_x = 0 & \text{in } \mathbf{R} \times (0, \infty) \\ u(x,0) = & \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases} \end{cases}$.