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PRELIMINARY EXAM IN DIFFERENTIAL EQUATIONS
AUGUST 2014

Time Limit: Four Hours.

Please write your 3-digit identification number on each page.

Proofs, or counter examples, are required for all problems.

1) Show that the problem (here $y = y(t)$)

$$y' = y^{2/3}, \quad y(0) = 0$$

has infinitely many solutions. Explain why the existence and uniqueness theorem does not apply here.

2) Assume that the function $u(x) \geq 0$ is continuous for $x \geq 1$, and for some number $K > 0$, we have

$$xu(x) \leq K + \int_1^x u(t)\,dt, \quad \text{for } x \geq 1\,.$$

Show that $u(x) \leq K$, for $x \geq 1$.

3) Let A be a real 3×3 constant matrix. Suppose that all solutions of

$$x' = Ax$$

are bounded as $t \to \infty$, and as $t \to -\infty$. Show that every solution is periodic, and there is a common period for all solutions.

Hint: Represent the solutions using the eigenvalues and the eigenvectors of A.

4) Show that $(0, 0)$ is the only rest point in $\mathbb{R}^2$ of the system

$$x' = 2y - x^3$$
$$y' = -x - y^5\,,$$

and prove that this rest point is asymptotically stable in Lyapunov's sense. (State Lyapunov's theorem that you are using.)

5) Find the solution to the differential equation

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} = 4u + x_1^3 & \text{in } (0,\infty)\times(0,\infty), \\ u(a,a) = a^3 & \text{for all } a > 0. \end{cases}$$

6) Suppose u is a harmonic function in $\mathbb{R}^2$, and has the values

$$u(0,0) = 1 \quad \text{and} \quad u(1,0) = 4.$$

Show that u is not strictly positive in the ball $\{|x| < 2\} \subset \mathbb{R}^2$.

7) Let $U \subset \mathbb{R}^n$ be an bounded open set. Suppose u has the properties

$$\begin{cases} \Delta(\Delta u) = 0 & \text{in } U, \\ \Delta u > 0 & \text{on } \partial U. \end{cases}$$

Show that u does not have a local maximum at any point $x \in U$, and therefore

$$\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x).$$

8) Let $U \subset \mathbb{R}^n$ be an bounded open set with smooth boundary.
Suppose u is a smooth solution of the differential equation

$$\begin{cases} u_t - \Delta u + u^3 = 0 & \text{in } U\times(0,\infty) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U\times(0,\infty) \\ u(x,0) = g(x) & \text{on } U\times\{t=0\} \end{cases}$$

Show that $\displaystyle\int_U |Du(x,t)|^2\,dx \le \int_U |Dg(x)|^2\,dx$ for all $t > 0$.