Preliminary Exam

Differential Equations

August 17, 2016

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ———-	Problem 2.1———
Problem 1.2 ———-	Problem 2.2———
Problem 1.3———	Problem 2.3———-
Problem 1.4———	Problem 2.4———

Part I total score : ———- Part II total score ———

Total score ———

Problem 1.1

Identify the stable and unstable subspaces for the linear ODE

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t). \; .$$

Problem 1.2

Write the following system in polar coordinates and determine if the origin is a center, a stable focus or an unstable focus.

$$\dot{x} = -y + xy^2, \quad \dot{y} = x + y^5.$$

Problem 1.3

Show that if a function $x(t)$ satisfies $0 \leq \frac{dx}{dt} \leq x^2$ for all t, and $x(0) = -1$, then $x(t) < 0$ for all $t \in [0, \infty)$.

Problem 1.4

Show that

$$\dot{x} = y, \quad \dot{y} = -x + (1 - x^2 - y^2)y$$

has a unique stable limit cycle which is the $\omega-$limit set of every trajectory except the critical point at the origin. (Hint: compute $\dot{r}$) .

Problem 2.1.

Find the solution to the differential equation $\begin{cases} u_x u_y = 1 & \text{in } (0,\infty)\times(0,\infty), \\ u(x,0) = 2\sqrt{x} & \text{for all } x > 0. \end{cases}$

Problem 2.2.

(i) Solve the following initial-value-problem.

$$\begin{cases} u_t + 2u_x - 4u_y + 5u_z = 0, \quad (x,y,z,t) \in \Re^3 \times (0,\infty), \\ \\ u(x,y,z,0) = x^2 + y - 3z \quad (x,y,z,t) \in \Re^3. \end{cases}$$

(ii) Use energy estimate method to show that the following initial-boundary-value problem admits at most one smooth solution $u = u(x,t)$.

$$\begin{cases} u_t + 2u_x + 12u_{xxx} = 0, \quad x \in (0,L),\ t \in (0,T) \\ u(x,0) = g(x),\ x \in (0,L), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0. \end{cases}$$

Problem 2.3.

Let U be the unit ball in $\mathbb{R}^3$. Suppose u solves the heat equation

$$\begin{cases} u_t - \Delta u = 0 \text{ in } U_T, \\ u(x,t) = 6t \text{ when } |x| = 1, \\ u(x,0) = g(x) \end{cases}$$

Suppose that $g \leq 0$. Prove that $u(x,t) \leq 6t + |x|^2$ for all $(x,t) \in U_T$.

Problem 2.4.

Find a solution formula for the following initial boundary value problem of the wave equation.

$$\begin{cases} u_{tt} - u_{xx} = 0, & x,\, t \in (0, \infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in (0, \infty), \\ u_x(0,t) = 0, & t \geq 0. \end{cases}$$