Preliminary Exam

Differential Equations

May 6, 2015

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ____________-	Problem 2.1__________
Problem 1.2 ____________-	Problem 2.2__________
Problem 1.3______________	Problem 2.3________-
Problem 1.4______________	Problem 2.4__________

Part I total score : ________- Part II total score ______________

Total score ______________

Problem 1.1

Solve the equation $\mathbf{x}'(t) = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \mathbf{x}(t)$ with initial value $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Problem 1.2

Suppose $\mathbf{x}(t)$ solves the equation $\mathbf{x}' = A\mathbf{x}$, with A being an $n \times n$ matrix, and its first coordinate is $x_1(t) = 5t^2 \cos(3t) + 2\sin(3t) - e^{2t}\sin(3t) + 4t$.

What is the minimum number of dimensions n?

Problem 1.3

Assume that $f(t)$ is continuous for $t \geq 0$, and let $F(t) = \frac{1}{t}\int_0^t f(s)\,ds$ be its average value. Show that if $f(t) \geq F(t)$ for all $t > 0$, then $F(t)$ is an increasing function, and also $f(t) \geq f(0)$ for all $t > 0$.

Problem 1.4

Consider the system

$$\frac{dx}{dt} = 5x - y + \frac{2}{3}xy$$
$$\frac{dy}{dt} = x + 3y$$

i) Determine all equilibria of the system.

ii) Classify the equilibrium at the origin as a source, sink, or saddle. What does this indicate about trajectories of the system?

Problem 2.1.

Find the solution to the differential equation $\begin{cases} u_{x_1} u_{x_2} = 1 + 4x_2 & \text{in } (0,\infty) \times (0,\infty), \\ u(a,0) = a & \text{for all } a \in \mathbb{R}. \end{cases}$

Problem 2.2.

Suppose $\Delta u = 0$ in the upper half-plane $\mathbb{R} \times (0, \infty)$, and suppose u is nonnegative.

i) Show that $u(0, y+h) \leq (1 + \frac{h}{y-h})^n u(0, y)$ for all $0 < h < y$.

ii) Show that $u(0, y) \leq y^n u(0, 1)$ for all $y \geq 1$.

(Hint: find an upper bound for $u_y(0, y)$.)

Problem 2.3.

Let $U \subset \mathbb{R}^n$ be an bounded open set. Suppose u solves the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u(x,t) = 4t - t^2 & \text{on } \Gamma_T \end{cases}$$

Prove that $u(x,t) < 4$ for all $(x,t) \in U_T$.

Problem 2.4.

Fix a number $L > 0$ and consider the initial-boundary value problem of the linear Boussinesq equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0 \ \text{ in } (0,L) \times (0,T), \\ u(x,0) = g(x) \text{ and } u_t(x,0) = h(x), \\ u(0,t) = 0, \ u(L,t) = 0, \ u_{xx}(0,t) = 0, \ \text{and } u_{xx}(L,t) = 0. \end{cases} \tag{*}$$

i) Define $E(t) = \int_0^L \left(u_t(x,t)^2 + u_x(x,t)^2 + u_{xx}(x,t)^2 \right) dx$. Show that

$$E(t) = \int_0^L \left(h^2(x) + (g'(x))^2 + (g''(x))^2 \right) dx$$

for any $0 \le t \le T$.

ii) Show that (*) admits at most one smooth solution..