Sample Questions for the PhD Preliminary in Differential Equations

Department of Mathematical Sciences
University of Cincinnati
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Ordinary Differential Equations

1. If the constant matrix A has a negative eigenvalue, then show the system $\dot{\mathbf{x}} = A\mathbf{x}$ has at least one nontrivial solution satisfying $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

2. (a) Give the definition of the norm $||\cdot||$ of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$.
   (b) If T is an invertible linear operator, show $||T^{-1}|| \geq 1/||T||$.
   (c) Use an example to show that, in general, $||T^{-1}|| \neq 1/||T||$.

3. Solve the system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \quad x(0) = x_0$. Determine the stable and unstable subspaces and sketch the phase portrait.

4. (a) State what it means for a function to satisfy a Lipschitz condition on a domain E.
   (b) Suppose $\mathbf{f} : E \to \mathbb{R}^n$ is Lipschitz on E. Show that $\mathbf{f}$ is uniformly continuous on E.

5. State and prove Gronwall's inequality.

6. Consider a pendulum of mass 1 on a rod of length 1 with units chosen so the acceleration of gravity is 1. Let $\theta(t)$ be the angle at time t (with $\theta = 0$ straight down). The dynamics of the pendulum may be modeled as $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ where $b \geq 0$ measures friction.

   (a) Rewrite the pendulum equation as a system of first-order equations using $v := \theta'(t)$. Find all equilibria (mod 2π in θ). Show that one of the equilibria is unstable.
   (b) Write down the *energy function* $E(\theta, v)$ as the kinetic energy $(mv^2/2)$ plus the potential energy (mgh). Show that $E(\theta, v)$ is a Liapunov function.

(c) Determine the stability of the other equilibrium.

7. Consider the system

$$\dot{x} = -y + x(r^4 - 3r^2 + 1)$$
$$\dot{y} = x + y(r^4 - 3r^2 + 1)$$

with $r^2 = x^2 + y^2$.

(a) Show $\dot{r} < 0$ on the circle $r = 1$ and $\dot{r} > 0$ on the circle $r = 2$. Use the only equilibrium is the origin to show there is a periodic orbit in the annular region $A_1 := \{\mathbf{x} \in \mathbb{R}^2 : 1 < |\mathbf{x}| < 2\}$.

(b) Show the origin is an unstable focus and use the Poincaré-Bendixon Theorem to show there is a periodic orbit in the annular region $A_2 := \{\mathbf{x} \in \mathbb{R}^2 : 0 < |\mathbf{x}| < 1\}$.

(c) Find all limit cycles of the system and classify them as stable, unstable, or neither.

8. Consider the system

$$\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2$$
$$\dot{y} = -y.$$

Determine the equilibria and their stability. Draw the bifurcation diagram. Draw the phase portraits for representative values of μ.

Partial Differential Equations

1. (i) Show that the boundary value problem for the bi-harmonic equation

$$\begin{cases} \Delta(\Delta u) = f(x) & \textit{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & \textit{on } \partial\Omega \end{cases}$$

has at most one solution. Here Ω is a bounded smooth region of R^n,

$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$$

is a given function, and ν is a unit normal vector at the boundary, pointing outside of Ω.

(ii) Show that uniqueness also holds for the problem

$$\begin{cases} \Delta(\Delta u) + \frac{\partial u}{\partial x_1} = f(x) & in\ \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & on\ \partial\Omega \end{cases}$$

2. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $\Omega_T = \Omega \times (0, T)$ and $c(x, t)$ be a bounded function in $\overline{\Omega}_T$.

   (a) Suppose $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ and satisfies

   $$\begin{cases} \Delta u + c(x,t)u - u_t \geq 0 & in\ \Omega_T \\ u(x,t) \leq 0 & in\ \Omega_T. \end{cases}$$

   Show that if $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \Omega_T$, then $u(x,t) \equiv 0$ for all $(x,t) \in \overline{\Omega} \times [0, t_0]$.

   (b) Suppose f is a C^1 function on R. Let $u,\ v \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$, and

   $$\begin{cases} u_t - \Delta u - f(u) \geq v_t - \Delta v - f(v) & in\ \Omega_T \\ u(x,0) > v(x.0) & for\ x \in \overline{\Omega} \\ u(x,t) > v(x,t) & for\ x \in \partial\Omega,,\ 0 < t < T. \end{cases}$$

   Show that $u(x,t) > v(x,t)$ for all $(x,t) \in \overline{\Omega}_T$.

3. Consider the initial-Boundary value problem of the KdV-Burgers equation posed on the finite interval $(0, 1)$;

$$\begin{cases} u_t + uu_x + u_{xxx} + u_x - u_{xx} = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = \phi(x), & x \in (0,1), \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0, & t \geq 0. \end{cases}$$

Show that

   (i) There exists at most one smooth solution.

   (ii) There exist constants $\gamma > 0$ and $C > 0$ such that its smooth solution (if exists) satisfies

   $$\int_0^1 u^2(x,t)dx \leq Ce^{-\gamma t} \int_0^1 \phi^2(x)dx$$

   for any $t \geq 0$.

4. Prove the following statements are true.

   (i) The limit of a uniformly convergent sequence of harmonic functions is harmonic.

   (ii) Let $\{u_n\}$ be a monotone increasing sequence of harmonic functions in a connected domain Ω and suppose for some point $y \in \Omega$, the sequence $\{u_n(y)\}$ is bounded. Then the sequence $\{u_n\}$ converges uniformly on any bounded domain $\Omega' \subset\subset \Omega$ to a harmonic function v.

5. Suppose u is a smooth solution of

$$u_t - u_{xx} + b(x)u_x + c(x)u = 0 \qquad in\ (a,b) \times (0,\infty),$$

and

$$u(a,t) = h_1(t), \qquad u(b,t) = h_2(t), \qquad u(x,0) = g(x)$$

where b, c, g, and f are continuous functions on the bounded interval (a,b), and $h_1(t)$ and $h_2(t)$ are non-positive continuous functions.

   (1) If $c(x) \geq 0$ and $g(x) \leq 0$ for any $x \in (a,b)$, then

$$u(x,t) \leq 0 \qquad for\ any\ (x,t) \in [a,b] \times [0,T].$$

   (Note: you are not allowed to use maximum principle directly.)

   (2) Prove that the conclusion of (1) is still true if $c(x)$ is assumed to be bounded instead of ≥ 0

6. Assume that

$$\begin{cases} \Delta u + C(x)u < 0 & in\ \Omega, \\ u \geq 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in R^n, with boundary $\partial\Omega$; $C(x)$ is a given continuous function. Prove that

$$u > 0 \qquad in\ \Omega.$$

Please note that no assumption is made on the sign of $C(x)$.

7. Suppose $c > 0$ is a given constant, and $f(x,t)$, $-\infty < t < \infty$, $x \in \Omega$ is a given function, where Ω is a bounded domain in R^n with smooth boundary. Prove that there can be only one bounded solution of the equation

$$\frac{\partial u}{\partial t} - \Delta u + cu = f$$

for $-\infty < t < \infty$ and $x \in \Omega$, with the boundary condition

$$u = \phi$$

on $\partial\Omega$. Here f and ϕ are given smooth functions.

8. Prove that $u \in C(R^n)$ is *harmonic* in R^n if it satisfies the mean value property, i.e.,

$$u(\xi) = \frac{1}{\omega_n} \int_{|x|=1} u(\xi + rx) dS_x,$$

where ω_n is the measure of the $(n-1)$ dimensional sphere in R^n.

9. Let $p = u_x,\ q = u_y$. Consider the first order nonlinear equation

$$u = p^2 - 3q^3,\ x \in R,\ y > 0,$$

with the initial condition $u(x,0) = x^2$. Solve $u = u(x,y)$ use the method of characteristics.