Linear Models Preliminary Exam

April 2013

1. Suppose Y_1, Y_2 and Y_3 are measurements of the angles of a triangle subject to error. The information is given as a linear model $Y_i = \theta_i + \epsilon_i$, where θ_i are the true angles, $i = 1,2,3$. Assume that $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2$ and ϵ_i's are independent. Obtain the least squares estimates of θ_i $i = 1,2,3$ (subject to the constraint $\sum_{i=1}^{3} \theta_i = \pi$).

2. 1. Let $\mathbf{X} \sim N_k(\boldsymbol{\Sigma}\theta, \sigma^2\boldsymbol{\Sigma})$, $r(\boldsymbol{\Sigma}) = k$, $\sigma^2 > 0$, θ fixed. Let $\mathbf{B} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathbf{1}\,(\mathbf{1}'\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1})^{-1}\mathbf{1}'\,\boldsymbol{\Sigma}^{-1}$, where $\mathbf{1}' = (1, \dots, 1)$.
   (a) Show that $\mathbf{B}$ is symmetric, $r(\mathbf{B}) = k - 1$, and $\mathbf{B}\boldsymbol{\Sigma}$ is idempotent.
   (b) Let $\mathbf{Y} = \mathbf{B}\mathbf{X}$. Find the distribution of $\mathbf{Y}$.
   (c) Obtain the distribution of $\mathbf{Y}'\boldsymbol{\Sigma}\mathbf{Y}$ when (i) $\theta = 0$ and (ii) $\theta \neq 0$.

3. Clearly state and prove the Gauss-Markov Theorem.

4. Suppose that Y_i is Poisson with mean μ_i, $g(\mu_i) = \alpha + \beta x_i$, where $g(.)$ is a link function, $X_i = 1$ for i = 1,…, n_A from group A and $X_i = 0$ for i = $n_A + 1, \dots, n_A + n_B$ from group B. Let μ_A and μ_B be the means of Group A and Group B, respectively. Find the fitted means $\hat{\mu}_A$ and $\hat{\mu}_B$ for any link function $g(.)$.

5. Let $X_1, X_2, \dots, X_n$ be a random sample from a multivariate normal distribution $N_p(\boldsymbol{\mu}, \Sigma)$. Construct the likelihood ratio test statistic for the hypothesis

   $$H0:\ \boldsymbol{\mu} = \boldsymbol{\mu}_0 \text{ against } H1:\ \boldsymbol{\mu} \neq \boldsymbol{\mu}_0.$$

   And show that the test based on the Hotelling's T^2 for the above hypothesis is equivalent to the likelihood ratio test.

6. A study is carried out on the effects of two types of incentives (factor A) on a person's ability to solve two types of problems (factor B). Twelve persons were randomly selected and assigned in equal numbers to the two incentive groups. The order of the two types of problems was then randomized independently for each person. The problem-solving ability scores are collected (the higher the score, the greater the ability to solve problems).
   (a) State an appropriate statistical model for this study and specify the model assumptions.
   (b) Complete the ANOVA table below.

Source	SS	DF	MS	EMS
A	975.38			
Subject(A)	148.75			
B	513.37			
A*B	155.04			
B.Subject(A)	34.08			
Total	1826.63			

   (c) Analysis the data and draw the conclusions based on the above ANOVA Table
   (d) Let $\overline{Y}_{.ij}$ represent the treatment mean under i^{th} incentive (factor A) and j^{th} type of problems (factor B). Assume it is known that $\overline{Y}_{.11} = 12.667$, $\overline{Y}_{.12} = 16.883$, $\overline{Y}_{.21} = 20.333$, $\overline{Y}_{.22} = 34.667$. The following comparisons are of interest:

$$L_1 = \mu_{.11} - \mu_{.12},\ L_2 = \mu_{.21} - \mu_{.22},\ L_3 = \mu_{.11} - \mu_{.21},\ L_4 = \mu_{.21} - \mu_{.22}$$

   Estimate these comparisons by means of confidence intervals. Use the Least Significant Difference with a pre-specified α. Clearly specify the degrees of freedom in the distributions involved.