UNIVERSITY OF Cincinnati

**Department of Mathematical Sciences**
PO Box 210025
Cincinnati OH 45221-0025

4th Floor French Hall West
Phone (513) 556-4050
Fax (513) 556-3417

# QUALIFYING EXAMINATION, MAY 2017 (4 HOURS)

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is $n$-dimensional Euclidean space. **Proofs, or counter examples, are required** for all problems.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with the property that for all $a, b \in \mathbb{R}$ with $a < b$ we have $\int_a^b f(x)dx \geq 0$. Prove that $f \geq 0$ on $\mathbb{R}$.

(2) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $5x_{n+1} = 3x_n + 4$. Prove that this sequence is convergent, and find its limit. **Hint:** First guess what the limit, if it exists, should be.

(3) For each positive integer $n$ let $f_n : (-1, 1) \to \mathbb{R}$ be a differentiable function with continuous derivatives and $f_n(0) = 0$. Suppose that there is a function $g : (-1, 1) \to \mathbb{R}$ such that $f_n' \to g$ uniformly in $(-1, 1)$. Prove that there is a function $f : (-1, 1) \to \mathbb{R}$ such that $f_n \to f$ uniformly and $f' = g$.

(4) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $p = (a, b) \in \mathbb{R}^2$. Suppose that there is an open set $U \subset \mathbb{R}^2$ with $p \in U$ such that $\frac{\partial f}{\partial x}(x, y)$ exists for all $(x, y) \in U$ and is continuous. Suppose in addition that $\frac{\partial f}{\partial y}(p)$ exists. Show that $f$ is differentiable at $p$. **Hint:** You might want to use the mean value theorem associated with functions of one variable.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{xy}{|x|+|y|} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = 0. \end{cases}$$

(a) Show that $f$ is continuous at $(0, 0)$.
(b) Is $f$ differentiable at $(0, 0)$? Justify your answer.

(6) Suppose that $V$ is an infinite dimensional vector space (over $\mathbb{R}$) with an inner product, and that $T : V \to V$ is an onto map, then its adjoint $T^*$ (that is, the map that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in V$) is injective.

(7) Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ are two collections of vectors from $\mathbb{R}^{2017}$ such that

$$\vec{v}_i \cdot \vec{w}_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set.

(8) Suppose that $V$ and $W$ are subspaces of the real vector space $\mathbb{R}^n$. If $V \cup W$ is a subspace of $\mathbb{R}^n$, prove that either $V \subseteq W$ or $W \subseteq V$.

*Date*: April 5, 2017.