UNIVERSITY OF Cincinnati

Department of Mathematical Sciences
PO Box 210025
Cincinnati OH 45221-0025

4^{th} Floor French Hall West
Phone (513) 556-4050
Fax (513) 556-3417

MATHEMATICS QUALIFYING EXAM
AUGUST 2015
Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space, $\mathbb{N}$ is the set of natural numbers (positive integers).

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. Let $(f_n)_1^\infty$ be a sequence of functions $[0,1] \xrightarrow{f_n} \mathbb{R}$.
(a) Define what it means to say that $(f_n)_1^\infty$ converges pointwise to a function f.
(b) Define what it means to say that $(f_n)_1^\infty$ converges uniformly to a function f.
(c) Suppose that $(f_n)_1^\infty$ converges uniformly to a continuous function f. Let $(x_n)_1^\infty$ be a sequence of real numbers that converges to $a \in [0,1]$. Prove that the sequence $(f_n(x_n))_1^\infty$ converges to $f(a)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ and suppose that $\alpha = \lim_{x\to\infty} f'(x)$ exists and is finite. Prove that $\alpha = 0$.

3. Suppose $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ are two sequences of real numbers, such that

$$a_{n+1} - a_n = |b_{n+1} - b_n| \text{ for } n = 0, 1, \ldots.$$

Use properties of real numbers to prove the following two implications.
(a) If $\{a_n\}$ is bounded then $\{a_n\}$ converges.
(b) If $\{a_n\}$ converges then $\{b_n\}$ converges.

4. The distance between two nonempty sets $A, B \subset \mathbb{R}^n$ is

$$\mathbf{d}(A,B) := \inf \{\|a - b\| : a \in A, b \in B\}.$$

(a) Suppose that $A, B \subset \mathbb{R}^n$ are disjoint, nonempty, A is compact, and B is closed. Prove that $\mathbf{d}(A,B) > 0$.
(b) Give an example of disjoint closed nonempty sets A and B in $\mathbb{R}^2$ with $\mathbf{d}(A,B) = 0$. (No formal proof is required for this part.)

5. Let $C[0,1]$ be the vector space of all continuous functions on the interval $[0,1]$. For each function $f \in C[0,1]$, define

$$(Tf)(x) := \int_0^x f(t)dt.$$

(That is, to each continuous function f we assign a new function $Tf : [0,1] \to \mathbb{R}$, which is given by the integral of f as shown.)

(a) Show that T is a linear mapping from $C[0,1]$ into $C[0,1]$.
(b) Prove that T is one-to-one.
(c) Show that T is not onto. Hint: *What property does each Tf have? (There is more than one property that can be used to answer this question!)*

6. Let $T : \mathbf{V} \to \mathbf{W}$ be a one-to-one linear transformation between vector spaces $\mathbf{V}$ and $\mathbf{W}$. Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\} \subset \mathbf{V}$. Prove that the following statements are equivalent:
(a) $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is linearly independent.
(b) $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$ is linearly independent.
Be sure to indicate where in your proof you use the one-to-one property.

7. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Prove that every matrix B with real entries such that $AB = BA$ has the form $B = sA + tI$ where s, t are real numbers and I is the identity matrix.

8. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable map with $F(1,1) = (1,0)$ and with derivative

$$DF(x,y) = \begin{pmatrix} y & x \\ -x & y \end{pmatrix} \text{ for all } (x,y).$$

(a) Prove that F is locally invertible at $(1,1)$, explaining what this means and citing the relevant theorem(s).
(b) Compute $DF^{-1}(1,0)$.
(c) Compute

$$\lim_{(x,y)\to(1,1)} \frac{\|F(x,y) - F(1,1)\|}{\|(x,y) - (1,1)\|}.$$

(Hint: *notice that $DF(x,y)$ is a scalar multiple of an orthogonal matrix*).