UNIVERSITY OF Cincinnati

MATHEMATICS QUALIFYING EXAM

MAY 2, 2016

Four Hour Time Limit

Notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space. The norm of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is $\|\mathbf{x}\| := \sqrt{\sum_{j=1}^n x_j^2}$ and the inner product is $\mathbf{x} \cdot \mathbf{y} := \sum_{j=1}^n x_j y_j$. $\mathcal{L}(V, W)$ is the linear space of all linear transformations from the vector space V to the vector space W.

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. Define the sequence $\{a_n\}_{n=0}^\infty$ by $a_0 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$, $n \geq 0$. Show that the sequence $\{a_n\}_{n=0}^\infty$ converges, and determine its limit.

2. Define a sequence of functions $\{f_n\}_1^\infty$ by $f_n(x) = \dfrac{1}{1 + n(nx-1)^2}$.
   (a) Show that $\{f_n\}_1^\infty$ converges pointwise to 0 on $[0, 1]$.
   (b) Verify that the convergence is **not uniform** on $[0, 1]$.

3. Show that for any integer $n \geq 1$, the equation $x^n + nx - 1 = 0$ has a unique positive solution x_n. Furthermore, show that x_n is such that for any $p > 1$ the series
$$\sum_{n=1}^\infty x_n^p$$
is convergent.

4. Define the notions of continuity and uniform continuity for a function $f : \mathcal{D} \to \mathbb{R}$ (Here $\mathcal{D}$ is a subset of $\mathbb{R}$). Use the definitions to prove that a continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous.

5. Consider the vector space V of continuous real-valued functions on the interval $[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. State and **prove** the Cauchy-Schwarz inequality for this inner product. (Hint: It may be easier to prove it first for the case when $\int_0^1 f^2(x)dx = 1$ and $\int_0^1 g^2(x)dx = 1$.)

6. Find the dimension and a basis for the linear space $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that preserves lengths, i.e., $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (a) Show also that T preserves orthogonality, i.e. if $\mathbf{x} \cdot \mathbf{y} = 0$, then $T(\mathbf{x}) \cdot T(\mathbf{y}) = 0$.
   (b) Show that the columns of the matrix of T in the standard basis of $\mathbb{R}^n$ are mutually orthogonal.

8. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$
Prove that
   (a) f is continuous;
   (b) the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exist;
   (c) f is not differentiable at $(0,0)$.