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MATHEMATICS QUALIFYING EXAM
MAY 2015
Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counterexamples, are required for all problems.

(1) Let $\mathbb{R} \xrightarrow{f} \mathbb{R}$ be differentiable. Suppose that f' is bounded on $\mathbb{R}$. Prove that there exist nonnegative constants C and D such that for all $x \in \mathbb{R}$, $|f(x)| \leq C|x| + D$.

(2) Define a sequence of functions $(f_n)_1^\infty$ by $f_n(x) := e^{-n(nx-1)^2}$.

(a) Show that $(f_n)_1^\infty$ converges pointwise to 0 on $[0, 1]$.

(b) Verify that the convergence is not uniform on $[0, 1]$.

(c) Prove that $\displaystyle\lim_{n\to+\infty} \int_0^1 f_n(x)\, dx = 0$.

(3) Let V be the span of $\mathbf{v} = (1, 2, 1) \in \mathbb{R}^3$. Let $W \subset \mathbb{R}^3$ be the orthogonal complement of V; that is, $W = \{\mathbf{x} \in \mathbb{R}^3 \colon \mathbf{x} \cdot \mathbf{v} = 0\}$. Let $P \colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation given by orthogonal projection onto W. Find the matrix that represents P relative to the usual basis for $\mathbb{R}^3$.

(4) Let V be a finite dimensional vector space. A linear operator T on V is *nilpotent* if $T \neq 0$ and for some $n \in \mathbb{N}$, $T^n = 0$.

(a) Let $W_i = T^i(V)$. Show that if $W_i \neq \{0\}$, then $W_{i+1} \subsetneq W_i$.

(b) Prove that there is a basis for V such that the matrix representing T is strictly upper triangular (that is, it is upper triangular with zeros on the main diagonal)

(5) A square matrix is called a *row stochastic matrix* if all its entries are non-negative real numbers and the entries in each row add up to 1. Prove that:

(a) The product of two row stochastic matrices is again a row stochastic matrix.

(b) Each stochastic matrix has 1 as an eigenvalue. (Hint: Exhibit an eigenvector.)

(6) Let $(X, \|\cdot\|)$ be a normed vector space and S be a non-empty subset of X. Which of the following conditions *always* implies the other two conditions? Prove one of the two implications. Also, give one example illustrating that one condition does not imply some other.

(a) S is closed and bounded.

(b) S is sequentially compact

(c) S is complete.

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(7) Let Ω be an open subset of $\mathbb{R}^n$. Suppose $\Omega \xrightarrow{f} \mathbb{R}$ is differentiable at the point $a \in \Omega$ with $f(a) \neq 0$. Prove that $1/f$ is differentiable at a, and find a formula for its derivative.

(8) Let $\mathcal{C}$ be the locus of all points (x, y, z) in $\mathbb{R}^3$ with

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 - y^2 - z = 0\,.$$

Show that $\mathcal{C}$ is a smooth curve in the following sense: For each point $p \in \mathcal{C}$, there exist open sets $U \subset \mathbb{R}$, $W \subset \mathbb{R}^3$ and a continuously differentiable map $f : U \to W$ such that $p \in W$ and $f(U) = \mathcal{C} \cap W$.