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MATHEMATICS QUALIFYING EXAM
APRIL 2014

Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) Let $[0,1] \xrightarrow{f} \mathbb{R}$ be a continuous function. Prove that the graph of f,

$$\mathbf{Gr}(f) := \{(x, f(x)) \mid x \in [0,1]\},$$

is a compact connected subset of $\mathbb{R}^2$.
(Hint: start by showing that the map $x \mapsto (x, f(x))$ is continuous.)

(2) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ converges. Prove that $(a_n)_{n=1}^{\infty}$ converges.

(3) Assume $[-1,1] \xrightarrow{f} \mathbb{R}$ is continuous on $[-1,1]$ and differentiable on $(-1,0) \cup (0,1)$. Suppose that $L := \lim_{x\to 0} f'(x)$ exists and is finite. Prove that f is differentiable at 0 with $f'(0) = L$.

(4) Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions $[0,1] \xrightarrow{f_n} \mathbb{R}$ that converges uniformly to a function $f : [0,1] \to \mathbb{R}$. Let $(b_n)_{n=1}^{\infty}$ be an increasing sequence of real numbers in $(0,1)$ that converges to 1. Prove that

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)\,dx = \int_0^1 f(x)\,dx\,.$$

(5) Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear operator satisfying $T \circ T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\operatorname{rank}(T) \le \frac{n}{2}\,.$$

(6) Let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a linear transformation between two vector spaces. Prove that T is injective if and only if it has the property that whenever S is a linearly independent subset of $\mathbb{V}$, $T(S)$ is a linearly independent subset of $\mathbb{W}$.

(7) Let $A := \begin{pmatrix} 8 & 2 \\ -8 & -2 \end{pmatrix}$. Find the entry in the first row and second column of A^{2014}.

(8) Suppose $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$ is differentiable at the origin with $Df(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0)|}{|(x,y)|} = 3\,.$$

Date: April 9, 2014.