ANALYSIS PRELIMINARY EXAMINATION, FALL 2017

Real Analysis
In this part of the exam, m, dx, dt denote the Lebesgue measure on $\mathbb{R}$.

(1) Let f be a monotone increasing absolutely continuous function on $[0,1]$. Prove that for every open set $\mathcal{O} \subset (f(0), f(1))$, the integral $\int_{f^{-1}(\mathcal{O})} f'(x)\,dx$ exists and $\int_{f^{-1}(\mathcal{O})} f'(x)\,dx = m(\mathcal{O})$. **Hint** *Notice that f needs not be strictly increasing.*

(2) Let $f \in L^1(\mathbb{R})$ and for every $x \in \mathbb{R}$ let $F(x) = \int_x^{x+1} f(t)dt$.
   (a) Prove that F is absolutely continuous on every interval $[a,b]$ and in particular F is measurable.
   (b) Prove that $F \in L^1(\mathbb{R})$. **Hint** *Tonelli.*
   (c) Compute $\int_R F(x)dx$.

(3) (a) Prove that if $f \geq 0$ is a Riemann integrable function on every interval $[0,a]$ with $0 < a < \infty$ and if the improper Riemann integral $\int_0^\infty f dx$ converges, then $f \in L^1([0,\infty))$ and its Lebesgue integral coincides with the improper Riemann integral.
   (b) Evaluate $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^n e^{-2x}dx$. **Hint** $1+y \leq e^y$ *for every* $y \in \mathbb{R}$.

(4) Let f_n be a sequence of Lebesgue measurable functions on $\mathbb{R}$ converging a.e. to a function f.
   (a) Assume that $0 \leq f_n(x) \leq f(x)$ a.e. Does it follow that $\int_{\mathbb{R}} f_n dm \to \int_{\mathbb{R}} f dm$? Either prove or find a counterexample. Does the answer change if we replace the condition that $f_n(x) \geq 0$ a.e., with the condition that f_n is integrable for all n?
   (b) Assume that $f_n(x) \geq f(x) \geq 0$ a.e.. Does it follow that $\int_{\mathbb{R}} f_n dm \to \int_{\mathbb{R}} f dm$? Either prove or find a counterexample. Does the answer change if the integral is over $[0,1]$ rather than $\mathbb{R}$?

Complex Analysis
In this part of the exam, $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. The unit circle centered at 0 is denoted $\mathbb{T}$. For complex-valued functions on a planar domain, the term "holomorphic" is the same as "analytic", and means a complex differentiable function.

1. Find a conformal map that maps the planar domain
$$\Omega := \{z \in \mathbb{C} : |z| > 1 \text{ and } \mathrm{Re}(z) > 0\}$$
to the unit disk $\mathbb{D}$.
2. For $0 \neq z \in \mathbb{C}$ let $f(z) = e^{1/z}$.
   (a) For $r > 0$ let γ_r be the counterclockwise oriented unit circle $\{z \in \mathbb{C} : |z| = r\}$ centered at 0. Show that the function $\varphi : (0,\infty) \to \mathbb{C}$ given by
$$\varphi(r) = \int_{\gamma_r} f(z)\,dz$$
is independent of r.
   (b) Show that $\lim_{z\to\infty} z\,f(z) = 1$.
   (c) Compute $\varphi(r)$ for $r > 0$.
3. Let Ω be a simply connected planar domain and $f : \Omega \to \mathbb{C}$ be a holomorphic function such that whenever γ is an anticlockwise oriented piecewise smooth loop in Ω we have
$$\int_\gamma \frac{f'(w)}{f(w)}\,dw = 0.$$
Show that there is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $e^{g(z)} = f(z)$ for each $z \in \Omega$. **Hint:** Fix a point $z_0 \in \Omega$ and consider the function $g : \Omega \to \mathbb{C}$ given by $g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)}\,dw$, where γ_z is any choice of a piecewise smooth curve in Ω with starting point z_0 and end point z. If you use this hint, you should also explain why this function g is independent of the choice of γ_z.
4. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, and suppose that $f(\mathbb{T}) \subset \mathbb{R}$.
   (a) Show that $f(0)$ is a real number.
   (b) Show that for each $z \in \mathbb{D}$ we must have $f(z) \in \mathbb{R}$.
   (c) Show that f must be a constant function.