Sample Questions for the PhD Preliminary Exam in Analysis

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Real Analysis

1. Let f be a real-valued function on $\mathbb{R}$. Show that the set of points where f is continuous is a G_δ set.

2. Let f be a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$. Suppose that, for every $n \in \mathbb{N}$,
$$\int_X [f]^n \, d\mu = \int_X f \, d\mu.$$
Show that $f = \chi_E$ a.e. on X, where E is a measurable subset of X.

3. Let f be absolutely continuous and strictly increasing on $[a, b]$. Show that for every open subset $\mathcal{O}$ of (a, b),
$$m(f(\mathcal{O})) = \int_{\mathcal{O}} f'.$$

4. Let f be a real-valued function that is integrable on $\mathbb{R}$ and let $\varepsilon > 0$. Show that there is a continuous function g that is identically zero outside some interval and such that
$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$
(Hint: Lusin's theorem.)

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and f a measurable real-valued function on X. Prove that
$$\int_X f^2 \, d\mu = 2 \int_0^\infty s \, \mu(\{x \in X : |f(x)| > s\}) \, ds$$

Complex Analysis

1. Let $\text{Log}(z)$ denote the principal branch of the logarithm function.
   (a) Show that in general $\text{Log}(ab) \neq \text{Log}(a) + \text{Log}(b)$.
   (b) For a given $a \in \mathbb{C}\backslash\{0\}$, determine the set of all z for which
   $$\text{Log}(az) = \text{Log}(a) + \text{Log}(z).$$

2. State and derive the Cauchy-Riemann equations.

3. Evaluate
   $$\int_{|z|=1} (z^2 + 2z) \csc(z)^2 dz.$$

4. Let $f(z) = z^2$.
   (a) Calculate $\int_0^{2\pi} f(2 + e^{i\theta}) d\theta$ and confirm it is non-zero.
   (b) Does Cauchy's theorem give $\int_{|z-2|=1} f(z)dz = 0$?
   Explain the seeming discrepency with (a).

5. Let $f(z) = u(z) + iv(z)$ be an entire function satisfying $u(z) \leq 0$ for all $z \in \mathbb{C}$.
   Show that $f(z)$ is constant.
   Hint: Consider $g(z) = e^{f(z)}$.

6. (a) Find the general form of any Mobius transformation which maps the unit disk D onto the upper half-plane.
   (b) What changes if we add the requirement that the origin should be mapped to i?

7. Show that $f(z) = z^4 - 3z^2 + 3$ has exactly one zero in the open first quadrant
   $Q_1 = \{z : \Re(z) > 0, \Im(z) > 0\}$.
   Hint: Use the Argument Principle.

8. Let R be a rational function. State and prove necessary and sufficient conditions for there to be a holomorphic branch of the logarithm of R in some domain Ω.