REAL & COMPLEX ANALYSIS PRELIMINARY EXAMINATION, SPRING 2013, MATHEMATICS, UNIVERSITY OF CINCINNATI

Real Analysis

(1) Let f be an absolutely continuous function on an interval I and let $E \subset I$ be such that $m(E) = 0$. Show that $m(f(E)) = 0$.

(2) Let μ be a measure on the measurable space $(\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Assume that there exists $K \geq 0$ such that

$$\int_{\mathbb{R}} e^{nx}\, d\mu(x) \leq K, \quad n = 1, 2, 3, ...$$

Prove that $\mu((0, \infty)) = 0$.

(3) Let $\{f_n\}$ be a sequence of non-negative Lebesgue measurable functions on $\mathbb{R}$ that converges pointwise to a function f that is Lebesgue integrable on $\mathbb{R}$.

$$\text{Show that if} \quad \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \quad \text{then} \quad \int_E f = \lim_{n\to\infty} \int_E f_n,$$

for any Lebesgue measurable set E.

(4) Let μ be a σ-finite measure on the measurable space $([0,1], \mathcal{M})$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$. For $x \in [0,1]$, define

$$F(x) = \mu([0,x]).$$

Show that μ is an absolutely continuous measure with respect to the Lebesgue measure m on $[0,1]$ if and only if F is an absolutely continuous function on $[0,1]$ satisfying $F(0) = 0$.

Complex Analysis

(1) (a) Suppose $\Omega \subset \mathbb{R}^2$ is a connected and open domain. Let $f_n : \Omega \to \mathbb{R}$ be a sequence of harmonic functions that converge uniformly to f. Prove that f satisfies the mean value property and hence is harmonic.
(b) Use part (a) to prove that not every continous function on a domain in $\mathbb{C}$ is the uniform limit of a sequence of complex polynomials.

(2) Suppose $f = u + iv$ is analytic in a domain D and $v(z) = [u(z)]^2$ for all $z \in D$. Show that f is constant in D.

(3) The fractional linear transformation F satisfies $F(0) = -1$, $F(2i) = \frac{1}{3}$, and $F(4i) = \frac{3}{5}$. Determine the image of the following sets of points under the function F.
  (a) The horizontal line $i + t$, $t \in \mathbb{R}$.
  (b) The circle of radius 1 centered at 1.

(4) Characterize the set of functions $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that are bounded away from 0 with $g(z) > |z|^{-\frac{7}{3}}$ for all $z \in \mathbb{C} \setminus \{0\}$.