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TOPOLOGY PRELIM EXAM, MAY 2023

To pass the prelim exam you should get at least 4 of the 5 questions solved correctly.

If you use a known theorem, or a similar result from a textbook, be sure to explicitly indicate so. Any other facts used must be proven. Proofs, or counter examples, are required for all problems.

(1) Prove that the set of rational numbers is not the intersection of a countable collection of open subsets of $\mathbb{R}$, where $\mathbb{R}$ is equipped with the Euclidean topology.

(2) Define a topology $\mathcal{T}$ on $\mathbb{R}$ by the rule that S is open if and only if S is empty or $\mathbb{R}\backslash S$ is finite.
   (a) Is $\mathcal{T}$ a topology?
   (b) Is $(\mathbb{R}, \mathcal{T})$ Hausdorff?
   (c) Is $(\mathbb{R}, \mathcal{T})$ compact?
   (d) Let $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$ be any polynomial function. Prove that f is continuous.

(3) Let X be a regular Hausdorff topological space and $A \subset X$. Let $Y = X/A$, the quotient space defined by the equivalence relation on X given by x_1 is equivalent to x_2 if and only if either $x_1 = x_2$ or $x_1, x_2 \in A$. Prove that Y is Hausdorff if and only if A is closed.

(4) Let X be a path-connected topological space and $x_0 \in X$. Let A be a closed subset of X such that $x_0 \in A$.
   (a) Give the definition of what it means to say that A is a deformation retract of X.
   (b) Suppose that A is a deformation retract of X. Prove that $\Pi_1(X, x_0)$ is group isomorphic to $\Pi_1(A, x_0)$.
   (c) Compute the fundamental group of the set $A \times [0, 1]$ where

$$A = ([0, 2] \times \{0, 1\}) \bigcup (\{0, 1, 2\} \times [0, 1]).$$

   Here the set is equipped with the Euclidean subspace topology. **Hint:** First draw a picture what this set looks like.

(5) Let $(X, \mathcal{T})$ be a connected, locally connected topological space, and $f : X \to \mathbb{R}$ be continuous. Prove that if $f(X)$ is a discrete subset of $\mathbb{R}$, then f is constant. Here by a discrete subset of $\mathbb{R}$ we mean that for each point z in that set there is an open (in the Euclidean topology) set $W \subset \mathbb{R}$ such that z is the only point from that subset belonging to W.