Statistics Part

1. Let $X_1, ..., X_n$ be iid $N(\mu_1, \sigma^2)$ and, independently, $Y_1, ..., Y_n$ be iid $N(\mu_2, \sigma^2)$.

   (a) Show that $(\bar{X}, \bar{Y}, S^2)$ formes a complete sufficient statistics for all three parameters, for a suitable statistic S^2.

   (b) Find the UMVUE for $(\mu_1 - \mu_2)/\sigma$

2. Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$, and let $S = \sqrt{S^2}$ where S^2 is the usual unbiased estimator of σ^2.

   Find the asymptotic normal distribution of S. Is it asymptotically efficient as an estimate of σ?

3. Let $X_1, ..., X_n$ be iid with pdf

   $$f(x|\theta) = \theta e^{-x\theta}, x > 0, \theta > 0$$

   Assume θ is assigned a prior dist

   $$\pi(\theta) = e^{-\theta}, \quad \theta > 0.$$

   (a) Find the posterior distribution of θ and identify it as being in the form of a commonly used distribution. Give the mean and variance of the posterior dist.

   (b) Find the Bayes rule with respect to the entropy loss function given by

   $$L(\theta, a) = a/\theta - log(a/\theta) - 1.$$

4. Let us denote the lognormal probability density function (pdf)

   $$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right]$$

   with $x > 0$, $\mu \in (-\infty, \infty)$, $\sigma \in (0, \infty)$. Suppose that $X_1, \ldots, X_m$ are identically and independently distributed (iid) positive random variables (rv) having the common pdf $f(x; \mu, 2)$ and that $Y_1, \ldots, Y_n$ are iid positive rv having the common pdf $f(y; 2\mu, 3)$. Also assume that the X's and Y's are independent. Here, μ is the unknown parameter and $m \neq n$. Describe the tests in their simplest implementable forms.

   (a) Given $\alpha \in (0, 1)$, find the Uniformly Most Powerful (UMP) level α test to choose between $H_0 : \mu = 1$ versus $H_1 : \mu > 1$.

   (b) Argue whether or not there exists a UMP level α test for deciding between $H_0 : \mu = 1$ versus $H_1 : \mu \neq 1$.

Probability Part

5. Let $\alpha > 0$ be a parameter to choose. Consider independent random variables $\{X_n\}_{n\in\mathbb{N}}$, each with distribution $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = n^{-\alpha}$. We are interested in the pattern 'three-heads-in-a-row': we say that there are three heads in a row at time k, if $X_k = X_{k+1} = X_{k+2} = 1$. Show that with probability one, three-heads-in-a-row shows up at most finitely many times, if $\alpha \in (1/3, \infty)$.

6. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with zero mean and finite second moment. Consider

$$X_{n,k} = \sum_{i=1}^{n} \frac{1}{2^{|i-k|}}\xi_i \quad \text{and} \quad S_n = X_{n,1} + \cdots + X_{n,n}, \quad n \in \mathbb{N}.$$

   (a) Find an estimate in the form of

$$\mathbb{E}S_n^2 \le Cn^\gamma, \text{ for all } n \in \mathbb{N},$$

   where C and γ are finite constants not depending on n. The constant C does not have to be optimal, but needs to be an explicit number.

   (b) Using the estimate obtained in part (a) above, prove that for all $\beta > 1/2$,

$$\frac{S_n}{n^\beta} \to 0 \text{ in probability as } n \to \infty.$$

7. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with cumulative distribution function $\mathbb{P}(X_1 \le x) = \left(\frac{x}{1+x}\right)^2$, $x \in [0, \infty)$. Show that

$$\sqrt{n}\left(\min_{i=1,\ldots,n} X_i\right)$$

   converges weakly as $n \to \infty$. Identify the limiting distribution.

8. Let $\{X_n\}_{n\in\mathbb{N}}$ be independent random variables with $\mathbb{P}(X_n = 1) = 1/n = 1 - \mathbb{P}(X_n = 0)$. Let $S_n := X_1 + \cdots + X_n$ be the partial sum.

   (a) Show that

$$\lim_{n\to\infty} \frac{\mathrm{Var}(S_n)}{\log n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{E}S_n - \log n}{\sqrt{\log n}} = 0.$$

   (b) Prove that

$$\frac{S_n - \log n}{\sqrt{\log n}} \Rightarrow \mathcal{N}(0,1)$$

   as $n \to \infty$. You may use the result in part (a) directly. Explain which central limit theorem you plan to use. State and verify all the conditions clearly.