Preliminary Examination - Multivariate Analysis

1:30-4pm, Monday, May 6, 2019

Answer all questions and show all your work.

1. The pressure drop measured across an expansion valve in a turbine is being studied. The design engineer considers the important variables that influence pressure drop reading to be gas temperature on the inlet side (A), operator (B), and the specific pressure gauge used by operator (C). These three factors are arranged in a factorial design, with A and B fixed, and C random. There are two replicates. The appropriate model for this design is

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

where τ_i, β_j, γ_k are the effect of A, B, and C, respectively.

a) State appropriate model assumptions and complete the ANOVA table. For E(MS), write the mathematical expressions, e.g., $E(MS_C) = \sigma^2 + an\,\sigma_{\beta\gamma}^2 + \cdots$.

Source	**SS**	**df**	**MS**	**F**	**E(MS)**
A	1023.36	2			
B	423.82	3			
C	7.19	2			
AB	1211.97				
AC	137.89				
BC	209.47				
ABC	166.11				
Error	770.50				
Total	3950.32	71			

b) Test the effects on A, AC and ABC factors, respectively. You only need to give the test statistics, and their associated sampling distribution and related degrees of freedom under the null hypothesis, respectively.

c) Now assume that the factor A is fixed effect, but B and C are random (the model form is unchanged.) Test the gas temperature effect, i.e., $H_0: \tau_i = 0$ for all i. You only need to give the test statistics, and their associated sampling distribution and related degrees of freedom under H_0.

2. Consider the one-way random effect model, i.e., $Y_{ij} = \mu + \tau_i + \epsilon_{ij}$ where Y_{ij} is the value of the response variable in the j^th^ trial for the i^th^ treatment, μ is fixed but unknown, τ_i's are independent $N(0, \sigma_\tau^2)$ and ϵ_{ij}'s are independent $N(0, \sigma^2)$ for $i = 1, \ldots, r$, $j = 1, \ldots, n$. And assume τ_i's and ϵ_{ij}'s are mutually independent. Let $\bar{Y}_{i.} = \sum_{j=1}^{n} Y_{ij}/n$ and $\bar{Y}_{..} = \sum_{i=1}^{r}\sum_{j=1}^{n} Y_{ij}/rn$. We can write the log-likelihood function as

$$l(\mu, \sigma_\tau^2, \sigma^2\,|y) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}r(n-1)\log\sigma^2 - \frac{r}{2}\log\lambda - \frac{SSE}{2\sigma^2} - \frac{SSTR}{2\lambda} - \frac{rn(\bar{Y}_{..} - \mu)^2}{2\lambda},$$

where $\lambda = \sigma^2 + n\,\sigma_\tau^2$.

a) Find the maximum likelihood estimators of $(\mu, \sigma_\tau^2, \sigma^2)$,

b) Find the restricted maximum likelihood (REML) estimators of σ_τ^2 and σ^2. First, show the restricted likelihood function of $(\sigma_\tau^2, \sigma^2)$ is

$$l_R(\sigma_\tau^2, \sigma^2|SSTR, SSE) = -\frac{1}{2}(rn-1)\log(2\pi) - \frac{1}{2}\log(rn) - \frac{1}{2}r(n-1)\log\sigma^2 - \frac{1}{2}(r-1)\log\lambda - \frac{SSE}{2\sigma^2} - \frac{SSTR}{2\lambda}.$$

Then, find the REML estimator of $(\sigma_\tau^2, \sigma^2)$.

3. Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be independent $N_p\ (\boldsymbol{\mu}_0, \boldsymbol{\Sigma}\ )$ random vectors with its pdf

$$f(\mathbf{x}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_0)\right\}$$

where $\boldsymbol{\mu}_0$ is a fixed $p \times 1$ vector, and $\boldsymbol{\Sigma}$ **is a** $p \times p$ positive definite (p.d.) matrix. Let $\overline{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i$.
Show that $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} (\boldsymbol{X}_i - \boldsymbol{\mu}_0)(\boldsymbol{X}_i - \boldsymbol{\mu}_0)^T$ is the maximum likelihood estimator of $\boldsymbol{\Sigma}$. [Use the following result: Given a $p \times p$ symmetric positive definite matrix $\boldsymbol{B}$ and a scalar $b\ (> 0)$, it follows that

$$\frac{1}{|\boldsymbol{\Sigma}|^b} \exp\left\{-\frac{1}{2} tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{B})\right\} \leq \frac{1}{|\mathbf{B}|^b} (2b)^{pb} \exp\{-bp\}$$

for all p.d. $\boldsymbol{\Sigma}_{p\times p}$, with equality holding only for $\boldsymbol{\Sigma} = \boldsymbol{B}/(2b)$.]

4. Let $\boldsymbol{X}$ be distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| > 0$. Show that $(\boldsymbol{X} - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu})$ follows the chi-square distribution with p degrees of freedom.

5. There is a random sample with size n from bivariate normal population with mean vector $\mu = \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}$ and var-cov matrix $\Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22}\end{bmatrix}$. For the $n = 21$ pairs of observations, we find that $\bar{\mathbf{x}} = \begin{bmatrix}.564 \\ .603\end{bmatrix}$, $S = \begin{bmatrix}.15 & .12 \\ .12 & .15\end{bmatrix}$ and $S^{-1} = \begin{bmatrix}18.52 & -14.81 \\ -14.81 & 18.52\end{bmatrix}$. Find the simultaneous intervals, i.e., T^2- intervals, for μ_1 and μ_2.