REAL ANALYSIS PRELIMINARY EXAM
MAY 2019

Time allowed: 2 hours 30 minutes.
Answer all problems and fully justify your work.
Throughout the exam m^n denotes Lebesgue measure on $\mathbb{R}^n$.

Question 1

Let $E := \cup_{n=1}^{\infty}(n - \frac{1}{2^n}, n + \frac{1}{2^n})$ and define the function $f\colon [0,\infty) \to \mathbb{R}$ by $f(x) := m([0,x] \cap E)$.

(1) Prove that f is an absolutely continuous function on the interval $[0,a]$ for any $a > 0$.
(2) Find the largest set on which the derivative f' exists. Compute f' when it exists.

Question 2

Recall that a measure space $(X, \mathcal{M}, \mu)$ is said to be finite if $\mu(X) < \infty$ and is σ-finite if there exists a sequence $X_n \in \mathcal{M}$ with $\mu(X_n) < \infty$ for all n and $\cup_{n=1}^{\infty} X_n = X$.

(1) Give an example of a measure space which is σ-finite but not finite.
(2) Give an example of a measure space which is not σ-finite.
(3) Prove that $(X, \mathcal{M}, \mu)$ is σ-finite if and only if there exists a function $f \in L^1(X, \mathcal{M}, \mu)$ such that $0 < f(x) < \infty$ for all $x \in X$.

Question 3

Let $f\colon [0,1] \to [0,\infty)$ be a Lebesgue measurable function. Show that

$$\lim_{n\to\infty} \int_0^1 \frac{f^n}{2+f^n}\,\mathrm{d}m = \frac{1}{3} m(\{x : f(x) = 1\}) + m(\{x : f(x) > 1\}).$$

Question 4

(1) Let $h\colon \mathbb{R} \to \mathbb{R}$ be continuous and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be measurable. Show that the composition $h \circ g\colon \mathbb{R}^2 \to \mathbb{R}$ is measurable.
(2) Let E be a Lebesgue measurable subset of $[0,1]$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := \sin(\pi y \chi_E(x))$. Explain why the integral $\iint_{[0,1]\times[0,1]} f \,\mathrm{d}m^2$ is defined and compute it.