Statistical Methods Prelim Exam

1:30-4pm, Thursday, May 9, 2019

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d $N(\theta, \sigma^2)$ where $\theta \in (\infty, \infty)$ is assumed unknown and σ^2 is known and fixed.
   (a) Derive the uniformly minimum variance unbiased estimator, say $T_h(\mathbf{X})$, for the parametric function $h(\theta) = e^\theta$.
   (b) Does $Var(T_h)$ attain the Cramer-Rao Lower Bound for unbiased estimators of $h(\theta) = e^\theta$? Show work to justify your answer.

2. Let X_i be independently distributed as $N(i\Delta, 1)$, $i = 1, \ldots, n$. Show that there exists the uniformly most powerful level α test of $H_0 : \Delta \leq 0$ against $H_1 : \Delta > 0$ where $\alpha \in (0, 1)$, determine it as explicitly as possible, and give the exact rejection region.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. from Beta(θ, θ) distribution with unknown $\theta > 0$. Let the prior distribution of θ be an exponential distribution, $p(\theta) = \frac{1}{\lambda} e^{-\theta/\lambda}$, $\theta > 0$, where λ is known. Under the square error loss, find the Bayes estimate and the posterior standard deviation of θ.

4. (a) Suppose that a random variable X has the logistic distribution with its pdf

$$f(x; \theta) = \frac{e^{-(x-\theta)}}{\{1 + e^{-(x-\theta)}\}^2}, \quad x \in (-\infty, \infty)$$

   where θ is the unknown parameter. Show that the Fisher information is $I_X(\theta) = 1/3$.

   (b) Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with the common pdf $f(x; \theta)$ in (a). Write down the likelihood equation in this situation. Is it possible to obtain an analytical expression of the maximum likelihood estimator (MLE), $\hat{\theta}_n = \hat{\theta}_n(\mathbf{X})$ where $\mathbf{X} = (X_1, \ldots, X_n)$? If not, how could one find the estimate $\hat{\theta}_n(\mathbf{x})$ for θ given the data $\mathbf{x}$?

   (c) Show that the MLE, $\hat{\theta}_n$, is consistent for θ.

   (d) Find the asymptotic distribution of $\sqrt{n}\{\hat{\theta}_n(\mathbf{X}) - \theta)\}$ as $n \to \infty$.