Complex Analysis Prelim Exam
UC Department of Math
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$\mathbb{C}$ is the field of complex numbers $z = x + iy$, $\mathbb{R}$ the field of real numbers, $\mathbb{D}$ the open unit disk, and $\hat{\mathbb{C}}$ the Riemann sphere (aka, extended complex plane)

If you use a known theorem, or a similar result from a textbook, be sure to explicitly indicate so. Any other facts used must be proven. Proofs, or counter examples, are required for all problems.

(1) Find the Laurent series of the function $\dfrac{1}{z(z-1)}$ for the region $2 < |z+2| < 3$.

(2) Let f and g be non-constant and holomorphic in $\mathbb{D} \setminus \{0\}$. Define $h(z)$ for $z \in \mathbb{D} \setminus \{0\}$ by $h(z) := f(z)g(z)$.
  (a) Explain why h has an isolated singularity at $z = 0$.
  (b) Discuss the nature of the isolated singularity $z = 0$ for h. (When is it: removable? a pole? an essential singularity?)

(3) Let $T(z) = \dfrac{z}{z+1}$.
  (a) Find the image $T(\hat{\mathbb{R}})$ where $\hat{\mathbb{R}}$ is the extended real line in $\hat{\mathbb{C}}$.
  (b) Find the image $T(K)$ where K is the unit circle $|z| = 1$.
  (c) Find the image $T(L)$ where L is the line $\mathfrak{Re}(z) = 1$.

(4) Let f be holomorphic in the annulus $A := \{1 < |z| < 2\}$. Suppose there is a sequence (p_n) of polynomials that converges locally uniformly in A to f. Prove that there is a function F that is holomorphic in $|z| < 2$ with $F|A = f$.

(5) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^4} dx$$

(6) How many roots does the polynomial $f(z) = z^5 + 2z^2 + 1$ have in the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$?