Notation

$X_n \xrightarrow{\mathcal{D}} X$ denotes convergence in distribution. $N(\mu, \sigma)$ denotes normal distribution with mean μ and variance σ^2.

Questions

1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common density function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

For each $n = 1, 2, \ldots$ let

$$Y_n = \begin{cases} 1, & \text{if } 0 < X_n < 1/2 \\ 0, & \text{otherwise} \end{cases}.$$

Let $S_n = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_n + Y_n$. Show that

$$\frac{S_n - n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma)$$

for an appropriate value of σ.

2. Let $Y_1, Y_2, \ldots$ be independent random variables such that for each $n = 1, 2, \ldots$

$$P(Y_n = 2^n) = e^{-n} \text{ and } P(Y_n = 0) = 1 - e^{-n}.$$

Show that the series $\sum_{n=1}^{\infty} Y_n$ converges with probability one.

3. Suppose X is a non-negative random variable with the property that there are constants $C_1, C_2 > 0$ and $\alpha > 0$, such that

$$C_1 x^{-\alpha} \le P(X > x) \le C_2 x^{-\alpha} \text{ for all } x > 1.$$

Discuss for what range of $p > 0$, we have $E(X^p) < \infty$.

4. State and prove Kolmogorov's maximal inequality for sums of independent random variables.