MATHEMATICS QUALIFYING EXAM

AUGUST 20, 2019

Four Hour Time Limit

Notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space. The norm of $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is $\|\vec{x}\| := \sqrt{\sum_{j=1}^n x_j^2}$ and the inner product is $\mathbf{x} \cdot \mathbf{y} := \sum_{j=1}^n x_j y_j$.

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. (a) For a function $f : [0, \infty) \to \mathbb{R}$, define what it means to say that $\lim_{x\to\infty} f(x) = a$ for some real a.

(b) Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous with $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Prove that there exists a $\xi \in [0, \infty)$ such that $f(\xi) = \max_{x\in[0,\infty)} f(x)$.

2. Let $(f_n)_1^\infty$ be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$.

(a) Define what it means to say that $(f_n)_1^\infty$ *converges uniformly on* $[0, 1]$ to $f : \mathbb{R} \to \mathbb{R}$.

(b) Suppose f_n, f are **strictly positive** functions (defined on $\mathbb{R}$), each f_n is continuous, and $(f_n)_1^\infty$ converges uniformly on $[0, 1]$ to f. Prove that $(1/f_n)_1^\infty$ converges uniformly on $[0, 1]$ to $1/f$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Suppose $f(0) = 0$, $f(1) = 0$, and that for all $x \in [0, 1]$, we have $f''(x) \le 0$. Prove that for all $x \in [0, 1]$, $f(x) \ge 0$.

4. Let (a_n) be a monotone decreasing sequence of positive numbers.

(a) Prove that $\sum_{n=1}^\infty a_n$ converges if and only if the series

$$\sum_{n=1}^\infty 2^n a_{2^n}$$

converges.

(b) Using the previous part, show that the following series diverge:

$$\sum_{n=1}^\infty \frac{1}{n}, \ \sum_{n=2}^\infty \frac{1}{n(\log n)}.$$

5. Consider the vector space $\mathbf{V}$ of 2×2 real symmetric matrices with the usual algebraic operations. (You do not need to verify that $\mathbf{V}$ is a linear space.) For $A, B \in \mathbf{V}$ define the inner product

$$\langle A, B\rangle := \operatorname{tr}(AB)$$

Here tr denotes the trace, i.e., the sum of the diagonal elements of a matrix.

(a) Verify that $\langle A, B\rangle$ is indeed an inner product on $\mathbf{V}$.

(b) Find an orthonormal basis of $\mathbf{V}$.

6. Let A, B be $n \times n$ matrices with real entries. Suppose that $AB = BA$ and that A has n distinct (real) eigenvalues.

(a) Prove that there is a basis $\mathcal{B}$ for $\mathbb{R}^n$ such that each vector in $\mathcal{B}$ is an eigenvector for both A and B.

(b) Show that there is an invertible $n \times n$ matrix P such that both PAP^{-1} and PBP^{-1} are diagonal matrices.

7. Let $\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{W} = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2\}$ in $\mathbb{R}^4$. (Note that $\vec{a}_1 \not\perp \vec{a}_2$.)

(a) Find vectors $\vec{w}$ in $\mathbf{W}$ and $\vec{v}$ in $\mathbf{W}^\perp$ so that $\vec{b} = \vec{w} + \vec{v}$.

(b) Find the distance from $\vec{b}$ to $\mathbf{W}$.

8. Show that the system of equations

$$\begin{aligned} x + y + z + w &= 4 \\ x^4 + y^3 + z^2 + w &= 4 \end{aligned}$$

can be solved uniquely for x, y in terms of z, w near $(1, 1, 1, 1)$. *Hint:* Verify the assumptions of the Implicit Function Theorem.