REAL ANALYSIS PRELIMINARY EXAM
AUGUST 2019

Time allowed: 2 hours 30 minutes.
Answer all problems and fully justify your work.
m^n is Lebesgue measure on $\mathbb{R}^n$ and m^* is Lebesgue outer measure on $\mathbb{R}$.

Question 1

Given any integrable function $\varphi\colon [0,1] \to \mathbb{R}$, define a corresponding sequence of functions by $\varphi_n(x) := n \int_x^{x+\frac{1}{n}} \varphi(t)\ \mathrm{d}t$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Here $\varphi(t) := 0$ for $t \in (1,2]$. With this definition, prove the following.

(1) If $f, g\colon [0,1] \to \mathbb{R}$ are integrable functions, then

$$\int_0^1 |f_n(x) - g_n(x)|\ \mathrm{d}x \leq \int_0^1 |f(x) - g(x)|\ \mathrm{d}x.$$

(2) If $g\colon [0,1] \to \mathbb{R}$ is continuous, then $g_n \to g$ uniformly.

(3) If $f\colon [0,1] \to \mathbb{R}$ is integrable, then $\int_0^1 |f_n - f|\ \mathrm{d}t \to 0$.
You may use without proof that for any $\varepsilon > 0$ there exists an continuous function $g\colon [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(x) - g(x)|\ \mathrm{d}x < \varepsilon$.

Question 2

(1) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq L|x - y|$ for every $x, y \in \mathbb{R}$. Prove that $m^*(f(A)) \leq Lm^*(A)$ for every set $A \subset \mathbb{R}$.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) = e^{-x^2}$. Show that if $A \subset \mathbb{R}$ with $m(A) = 0$ then $m(f(A)) = 0$.

(3) Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. For sets $A \subset \mathbb{R}$, is it necessarily true that $m(A) = 0$ implies $m(g(A)) = 0$?

Please turn over for remaining questions.

Question 3

Let $f_n\colon [0,1] \to \mathbb{R}$ be absolutely continuous functions with $f_n(0) = 0$. Suppose there is integrable $g\colon [0,1] \to \mathbb{R}$ such that $\int_0^1 |f_n'(t) - g(t)|\,\mathrm{d}t \to 0$ as $n \to \infty$.

Prove that f_n converge uniformly to some absolutely continuous function $\phi\colon [0,1] \to \mathbb{R}$ which satisfies $\phi'(x) = g(x)$ for Lebesgue almost every x.

Question 4

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces. Give the definition of the product measure $\mu \times \nu$ and the family of sets on which it is defined.

Let $L = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \subset \mathbb{R}^2$ and $\Delta = \{(x,x)\colon x \in \mathbb{R}\} \subset \mathbb{R}^2$.

(1) Find with proof the product measure $(m \times m)(L)$.
(2) Find with proof the product measure $(m \times c)(L)$.
(3) Find with proof the product measure $(m \times m)(\Delta)$.
(4) Find with proof the product measure $(m \times c)(\Delta)$.

Here c is counting measure defined on all subsets of $\mathbb{R}$ (so $c(A)$ is the number of elements of a set A).