Topology Preliminary Examination August 2019
University of Cincinnati Department of Mathematical Sciences

Show all essential work.

1. Let X be a compact metric space and $\{U_\alpha, \alpha \in A\}$ an open cover of X. Prove that there is a $\rho > 0$ such that if $d(x, y) < \rho$ then there exists an $\alpha \in A$ so that $x, y \in U_\alpha$.

2. Let X be a space that is the union of subspaces $S_1, S_2, \ldots, S_n$, each of which is homeomorphic to the unit circle. Assume there is a point p of X such that $S_i \bigcap S_j = \{p\}$ for $i \neq j$.

   (a) Show that X is Hausdorff if and only if each space S_i is closed in X.

   (b) Give an example to show that X need not to be Hausdorff.

   (c) Assume that X is Hausdorff. Determine $\pi_1(X, p)$. Justify your answer.

3. Let $f : A \to \prod_{\alpha \in j} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α.

   (a) Let $\prod X_\alpha$ have the product topology. Show that if f is continuous then each function f_α is continuous.

   (b) Let $\prod X_\alpha$ have the product topology. Show that if each f_α is continuous then f is continuous.

   (c) Let $\prod X_\alpha$ have the box topology. Give an example to show that if each f_α is continuous then f need not be continuous.

4. Show that a path connected space is connected. Show that if $U \subset \mathbb{R}^n$ is open and connected then it is path connected.

5. Let $B \subset \mathbb{R}^3$ be the closed unit ball, so that

$$B = \{X = (x, y, z) : |X|^2 = x^2 + y^2 + z^2 \leq 1\}.$$

   Define an equivalence relation on B by $X \sim Y$ if $|X| = |Y|$. Show that $B/\sim$ is homeomorphic to $[0, 1]$.

   Recall that $X/\sim$ is the collection of equivalence classes determined by $\sim$. Introduce $q : X \to X/\sim$ as the function that assigns each point of X to the equivalence class to which it belongs. The topology on $X/\sim$ is defined by saying $A \subset X/\sim$ is open if and only if $q^{-1}(A)$ is open in X, i.e., the union of the equivalence classes represented by the pts in A is open in X.