PhD Preliminary Exam in Algebra - Fall 2020

Full marks may be obtained by complete answers to 4 questions.
Time allowed - 2 1/2 hours
No calculators

(1) Let F be a field with 2^{16} elements. Find all the subfields of F, giving complete proofs of all your claims.

(2) Prove that there exists an abelian Galois extension over $\mathbb{Q}$ of degree 11. Write it in the form $\mathbb{Q}(\quad)$ for $\quad \in \mathbb{C}$. (Hint: Take a suitable cyclotomic field, and look at its subfields.)

(3) Consider the ring $\mathbb{Z}[X]$. Describe the irreducible elements of $\mathbb{Z}[X]$. Show that $\mathbb{Z}[X]$ is a unique factorization domain.

(4) Let $K \subset L$ be a finite extension of fields.
  (a) Define what it means for the extension to be *separable*.
  (b) Suppose that $[L : K] = 2$. Show that the extension is not separable if and only if the characteristic of K is 2 and $L = K(\quad)$ where $\quad^2 \in K$.

(5) Let $K \subset L$ be a finite extension of fields.
  (a) Define what it means for the extension to be *normal*.
  (b) Define what it means for L to be a splitting field for a polynomial $f(x) \in K[x]$.
  (c) Prove that the extension is normal if and only if L is a splitting field for a polynomial $f(x) \in K[x]$.