## Ordinary Di erential Equations Preliminary Exam August 2020

1. Find and classify the fixed points for the system

$$\dot{x} = -10x + 10y$$
  
$$\dot{y} = 28x - y - xz$$
  
$$\dot{z} = \frac{8}{3}z + xy.$$

2. Let 
$$J := \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$
 be a Jordan block. Show that
$$e^{Jt} = e^{-2t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- 3. A. Define what it means for a function f to be Lipschitz on a domain E.B. Prove that if f is Lipschitz on a domain E then f is uniformly continuous on E.
- 4. Consider the system

$$x' = x - y - x^{3}$$
$$y' = x + y - y^{3}.$$

A. Show that the origin (0,0) is the only rest point.

Hint: Show that the curves  $x - y - x^3 = 0$  and  $x + y - y^3 = 0$  only intersect at the origin.

B. Show that the system has a limit cycle.

Hint: Compute the scalar product of  $\langle x - y - x^3, x + y - y^3 \rangle$  and  $\langle x, y \rangle$ . Switch to polar coordinates and show the annulus  $\rho^2 \leq x^2 + y^2 \leq R^2$  is a trapping region for  $\rho > 0$  sufficiently small and R > 0 sufficiently large.

5. Show that the rest point (0,0) is asymptotically stable for

$$\begin{aligned} x' &= y - x^3 \\ y' &= -x - y^5. \end{aligned}$$

Suppose that  $x(0) = x_0$  and  $y(0) = y_0$ . What can you say about the behavior of the solution (x(t), y(t)) as  $t \to \infty$ ?