## Ordinary Di erential Equations Preliminary Exam August 2020

1. Find and classify the fixed points for the system

$$
\begin{aligned}
\dot{x} & =-10 x+10 y \\
\dot{y} & =28 x-y-x z \\
\dot{z} & =\frac{8}{3} z+x y .
\end{aligned}
$$

2. Let $J:=\left[\begin{array}{cccc}-2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2\end{array}\right]$ be a Jordan block. Show that

$$
e^{J t}=e^{-2 t}\left[\begin{array}{cccc}
1 & t & \frac{1}{2} t^{2} & \frac{1}{3!} t^{3} \\
0 & 1 & t & \frac{1}{2} t^{2} \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{array}\right]
$$

3. A. Define what it means for a function $f$ to be Lipschitz on a domain $E$.
B. Prove that if $f$ is Lipschitz on a domain $E$ then $f$ is uniformly continuous on $E$.
4. Consider the system

$$
\begin{aligned}
& x^{\prime}=x-y-x^{3} \\
& y^{\prime}=x+y-y^{3} .
\end{aligned}
$$

A. Show that the origin $(0,0)$ is the only rest point.

Hint: Show that the curves $x-y-x^{3}=0$ and $x+y-y^{3}=0$ only intersect at the origin.
B. Show that the system has a limit cycle.

Hint: Compute the scalar product of $\left\langle x-y-x^{3}, x+y-y^{3}\right\rangle$ and $\langle x, y\rangle$. Switch to polar coordinates and show the annulus $\rho^{2} \leq x^{2}+y^{2} \leq R^{2}$ is a trapping region for $\rho>0$ sufficiently small and $R>0$ sufficiently large.
5. Show that the rest point $(0,0)$ is asymptotically stable for

$$
\begin{aligned}
x^{\prime} & =y-x^{3} \\
y^{\prime} & =-x-y^{5} .
\end{aligned}
$$

Suppose that $x(0)=x_{0}$ and $y(0)=y_{0}$. What can you say about the behavior of the solution $(x(t), y(t))$ as $t \rightarrow \infty$ ?

