Preliminary Exam

Partial Differential Equations

August 12, 2020

Name: **Student Id #:**

Instruction: *Choose only five (out of six) problems to do. If you have tried all six problems, please indicate which five problems to be count.*

Score:

Problem 1 ————————- Problem 2——————

Problem 3 ————————- Problem 4 ——————

Problem 5 ————————- Problem 6 ——————

Total score ——————————

Problem 1 Using the method of characteristics to find the solution to

$$\begin{cases} u_t + 2u_x = u^2 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = \cos x & \text{on } \mathbb{R}. \end{cases}$$

Is the solution you obtained globally defined? If not, at what time(s) does it blow up?

Problem 2 Find the entropy solution to the problem

$$u_t + 8u^3 u_x = 0, \quad u(x,0) = g(x), \quad x \in \mathbb{R}, \quad t > 0.$$

where

(a)

$$g(x) = \begin{cases} 2 & if\ x < 0, \\ -1 & if\ x \geq 0. \end{cases}$$

(b)

$$g(x) = \begin{cases} -1 & if\ x < 0, \\ 2 & if\ x \geq 0. \end{cases}$$

Problem 3: For $x = (x_1, x_2) \in \mathbb{R}^2$ and $u(x) = \log(x_1^2 + x_2^2)$, show that

$$-\int_{\mathbb{R}^2} u \Delta \phi dx = 4\pi\phi(0)$$

for any $\phi \in C_0^\infty(\mathbb{R}^2)$.

Hint: $\int_{\mathbb{R}^2} u\Delta\phi dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon} u\Delta\phi dx$ where $B_\epsilon = \{x \in \mathbb{R}^2 : |x| \leq \epsilon\}$.

Problem 4: Show that

$$u(x_1, x_2) := \log(\log(x_1^2 + x_2^2))$$

is the unique (smooth) solution of the nonlinear boundary value problem

$$\Delta v + |\nabla v|^2 = 0 \ in \ \Omega, \quad v = u \ on \ \partial\Omega$$

where $\Omega \subset \mathbb{R}^2$ is a bounded regular domain such that, $\overline{\Omega} \subset \mathrm{dom}(u) = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 > 1\}$.

Problem 5: Let $T > 0$ be a given terminal time, $L > 0$, and h be a continuous function on $[0, L]$. Consider the following initial-boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & (x,t) \in (0,L) \times (0,T] \\ u(0,t), \quad u(L,t) = 0 & t \in [0,T] \\ u(x,0) = 0, \quad u_t(x,0) = h(x) & x \in (0,L) \end{cases} \qquad (*)$$

(i) Suppose that $u = u(x,t)$ is a smooth solution of $(*)$. Show that for each $t > 0$,

$$\int_U \frac{1}{2}\left[u_t(x,t)\right]^2 + \frac{1}{2}|u_x(x,t)|^2 \,\mathrm{d}x \leq \int_U \frac{1}{2}[h(x)]^2 \,\mathrm{d}x.$$

(ii) Prove that $(*)$ has at most one smooth solution. (*Hint: Use the result from the previous part.*)

Problem 6: Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded, and connected with a smooth boundary $\partial\Omega$. Let $T > 0$ be a terminal time and recall the definitions of the parabolic cylinder $\Omega_T := \Omega \times (0, T]$ and the parabolic boundary $\Gamma_T := \overline{\Omega}_T \setminus \Omega_T$. Assume that $u = u(x, t)$ with $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ \quad u(x,t) = 4(t-1) - t^2 & \text{on } \Gamma_T \end{cases}$$

Prove that $u(x, t) \leq 0$ for all $(x, t) \in \Omega_T$.